



The $N+1$ Queens Problem

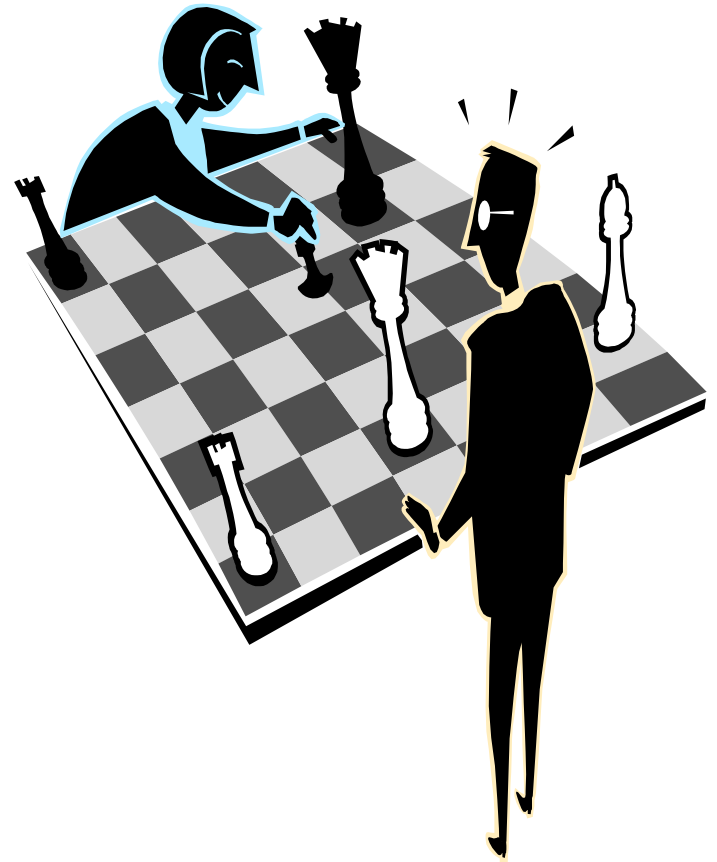
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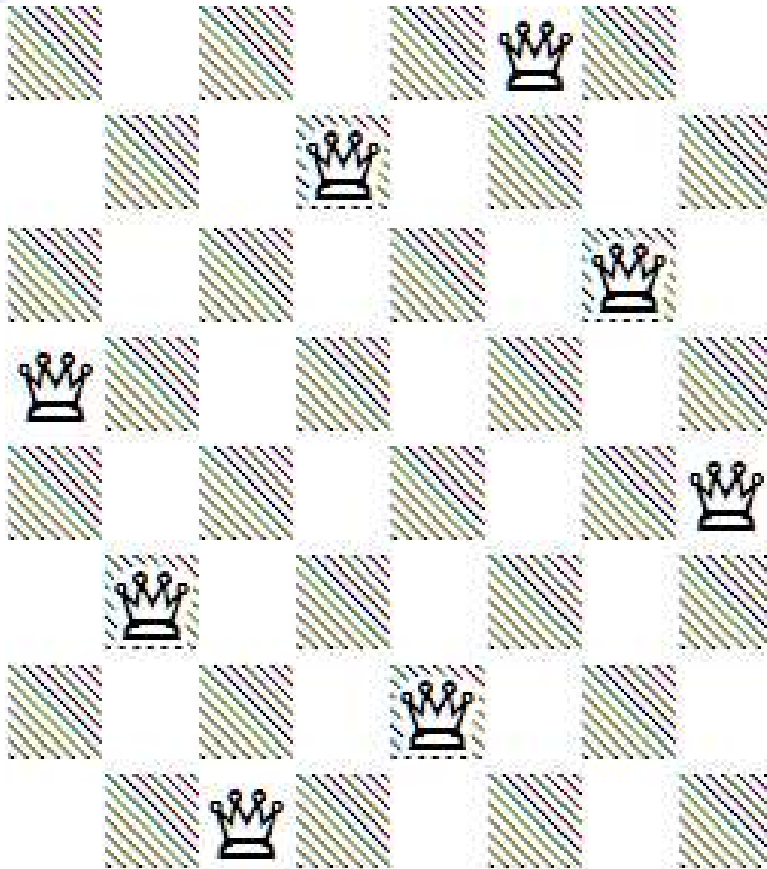
Last update: July 12, 2005

Acknowledgements

- Joint work with Gerd Fricke and R. Duane Skaggs



The Eight Queens Problem



- Place eight queens on a standard chessboard so that no two attack each other.
- First posed in 1848.
- Generalized in 1850 to N queens on an $N \times N$ board.



Counting N-queens solutions

N	Number of solutions	N	Number of solutions
4	2	11	2,680
5	10	12	14,200
6	4	13	73,712
7	40	14	365,596
8	92	15	2,279,184
9	352	16	14,772,512
10	724	17	95,815,104



More counting

N	Number of solutions to N-queens problem
18	666,090,624
19	4,968,057,848
20	39,029,188,884
21	314,666,222,712
22	2,691,008,701,644
23	24,233,937,684,440
24	227,514,171,973,736



More about N-queens...

- Theorem: For $N > 3$, there is at least one solution to the N-queens problem.
- Proved first by Ahrens in 1910.
- Also proved by Hoffman, Loessi, and Moore in 1969. (Mathematics Magazine, March-April 1969, 66-72.)
- Also proved by others.



At most N queens, right?

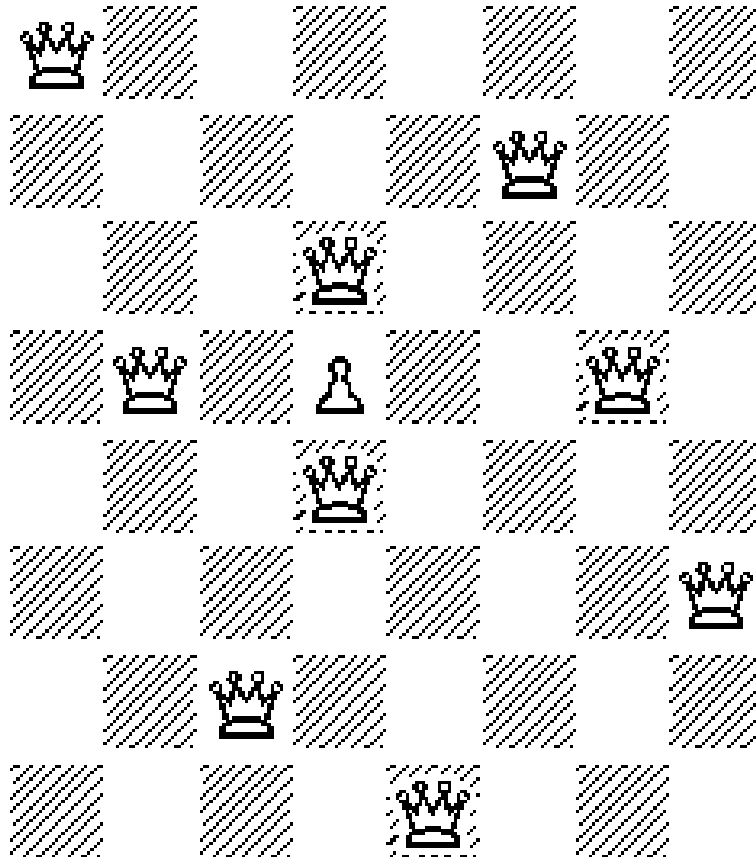
- **Clearly**, each row can only have one queen.
- **Clearly**, an $N \times N$ chessboard can have at most N non-attacking queens placed on it.



The Nine Queens Contest

- January-March 2004, The Chess Variant Pages at chessvariants.org.
- If we place a pawn between two queens on the same row (or column or diagonal), the queens no longer attack each other.
- Question: How many pawns do we need in order to put 9 nonattacking queens on a standard chessboard?

Solution to 9 Queens Contest



- Answer: One pawn.
- A solution:
 - QUEENS at a8, b5, c2, d4, d6, e1, f7, g5, h3
 - PAWN at d5



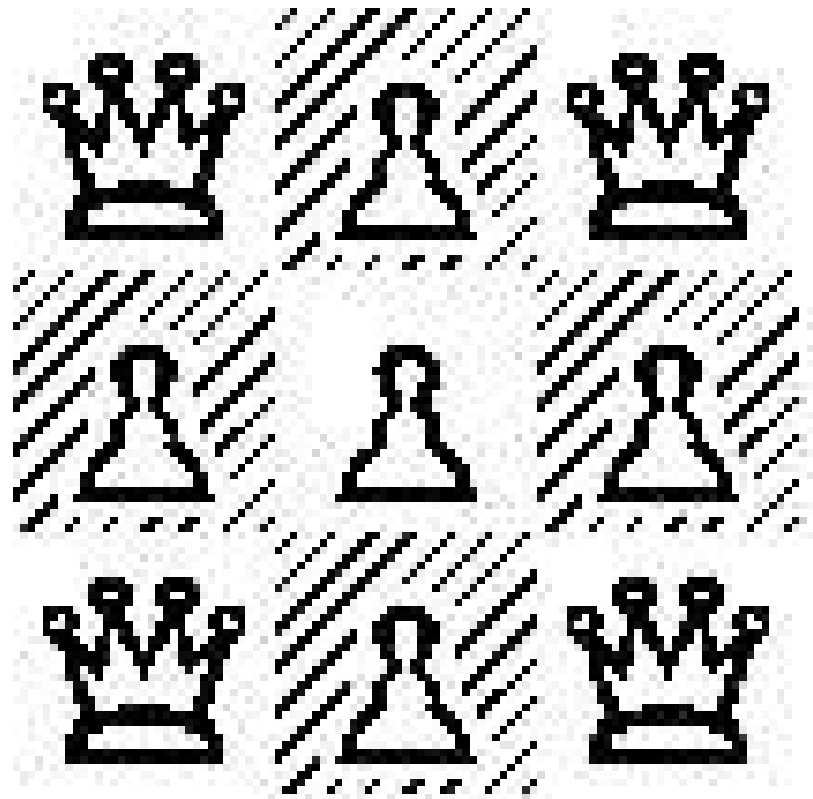
The $N+k$ Queens Problem

- Given $N+k$ queens, how many pawns need to be put on an $N \times N$ board to allow the queens to be placed on the board so they don't attack each other?
- This is the **queens' separation number** $s_Q(N+k, N)$.

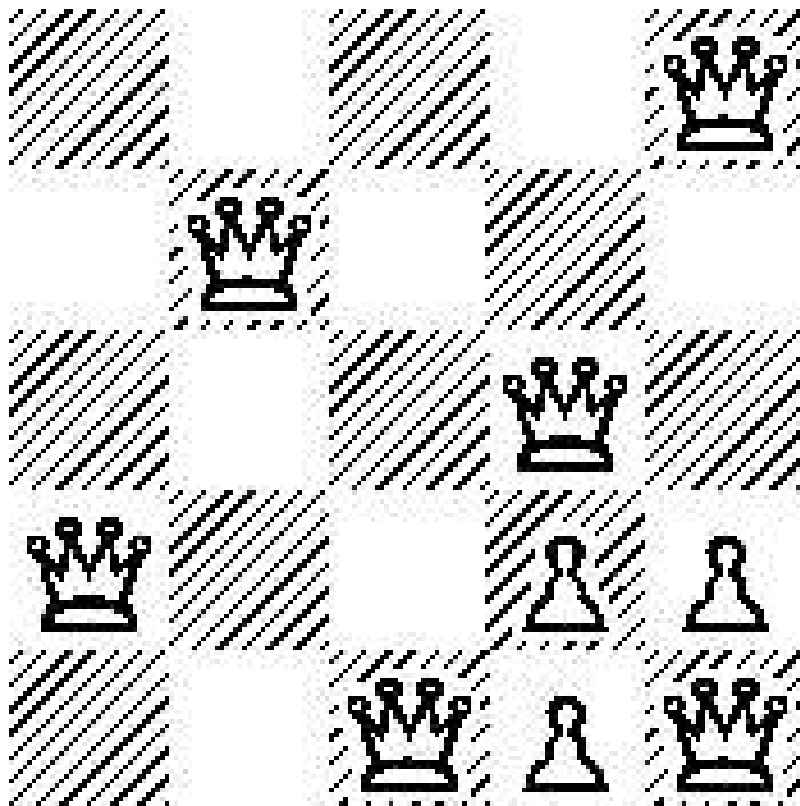


The N+1 Queens Problem

- $s_Q(4,3) = 5$.
- $s_Q(5,4)$ doesn't exist.
 - When we put 5 queens on a 4 x 4 board, at least two queens will be on adjacent squares.



The N+1 Queens Problem



- $s_Q(6,5) = 3$.
 - Verified by computer search.
- Theorem: For $N \geq 6$, $s_Q(N+1, N) = 1$

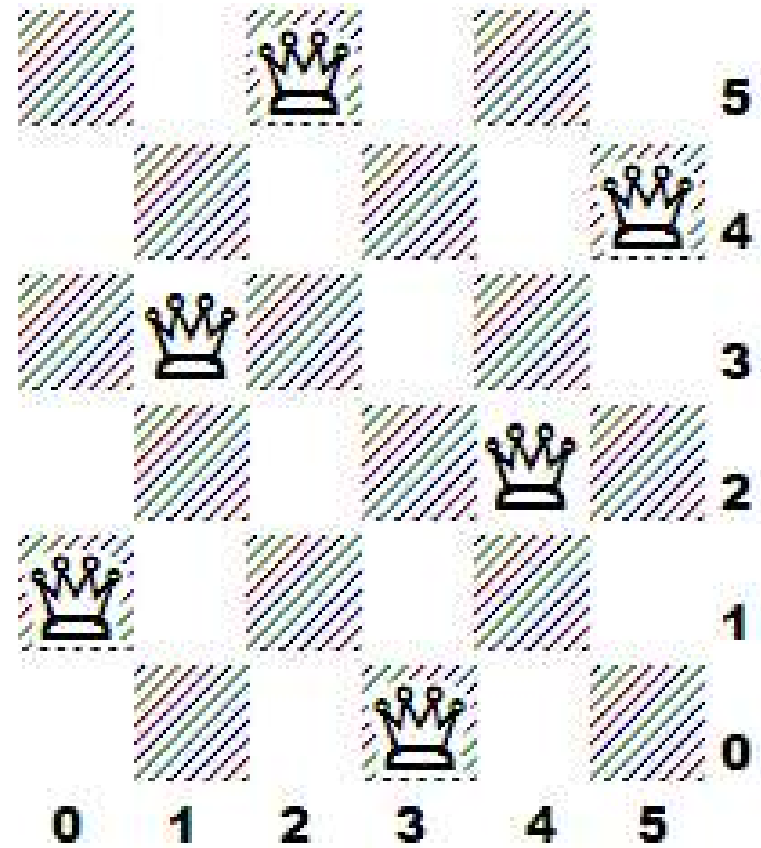


Sketch of Proof of Theorem

- We take known solutions to the n-queens problem and add extra rows, columns, queens, and a pawn.
- We have four main patterns and a few exceptional cases.

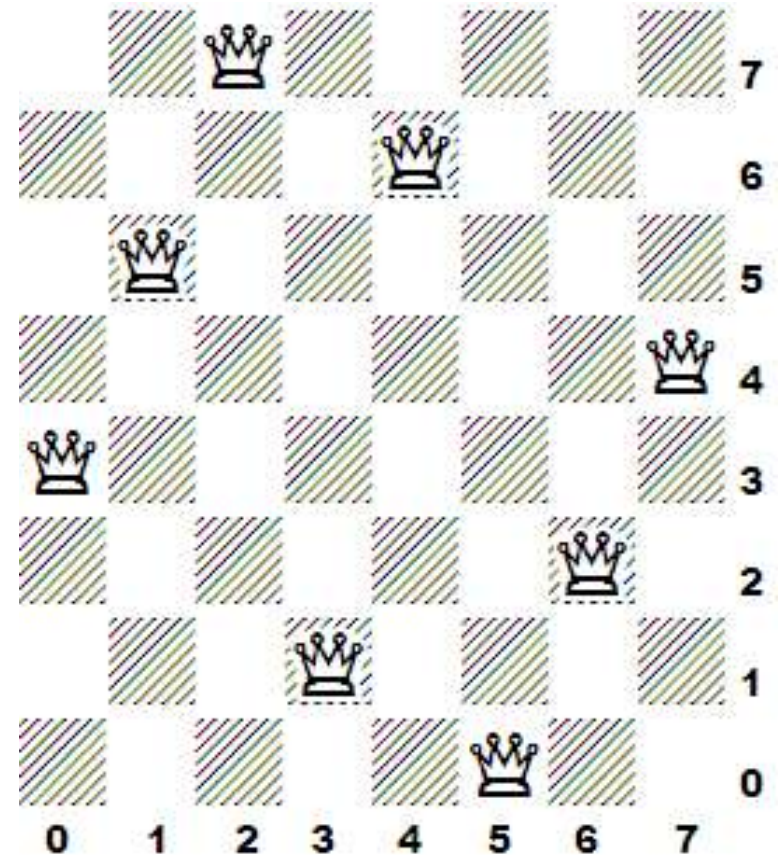
N-Queens Construction A

- $n \geq 4$
- $n \equiv 0$ or $4 \pmod{6}$
- Number rows and columns $0, 1, \dots, n-1$
- Queens at $(2i+1, i)$ for $i=0, \dots, n/2-1$
- Queens at $(2i-n, i)$ for $i=n/2, \dots, n-1$.



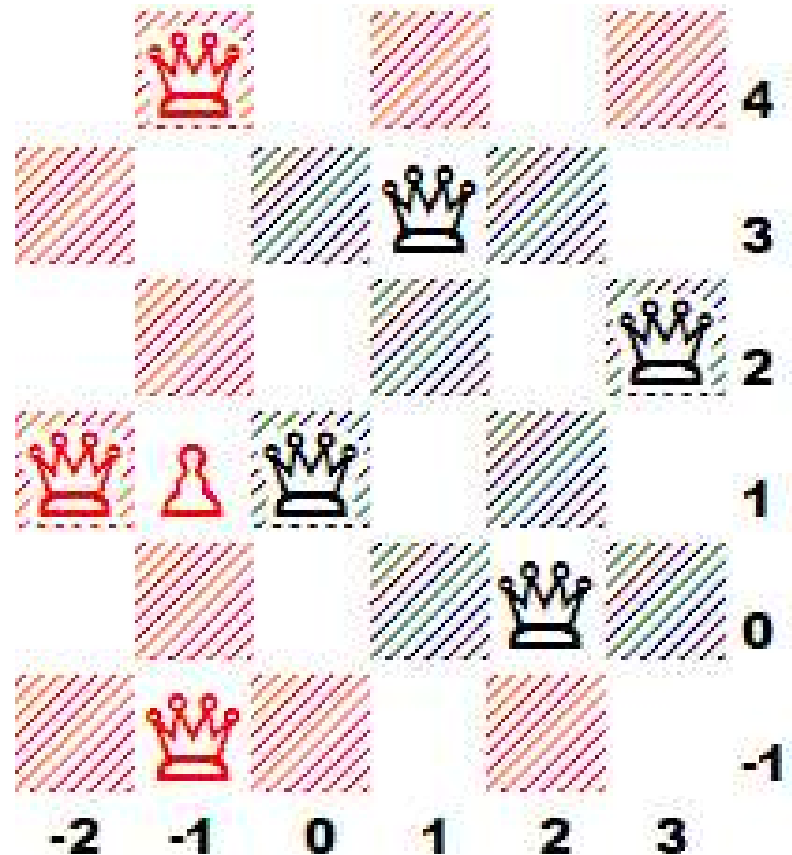
N-Queens Construction B

- $n \geq 4$, $n \equiv 2$ or $4 \pmod{6}$
- Number rows and columns $0, 1, \dots, n-1$
- Queens at $(n/2+2i-1 \pmod{n}, i)$ for $i=0, 1, \dots, n/2-1$ and at $(n/2+2i+2 \pmod{n}, i)$ for $i=n/2-1, \dots, n-1$



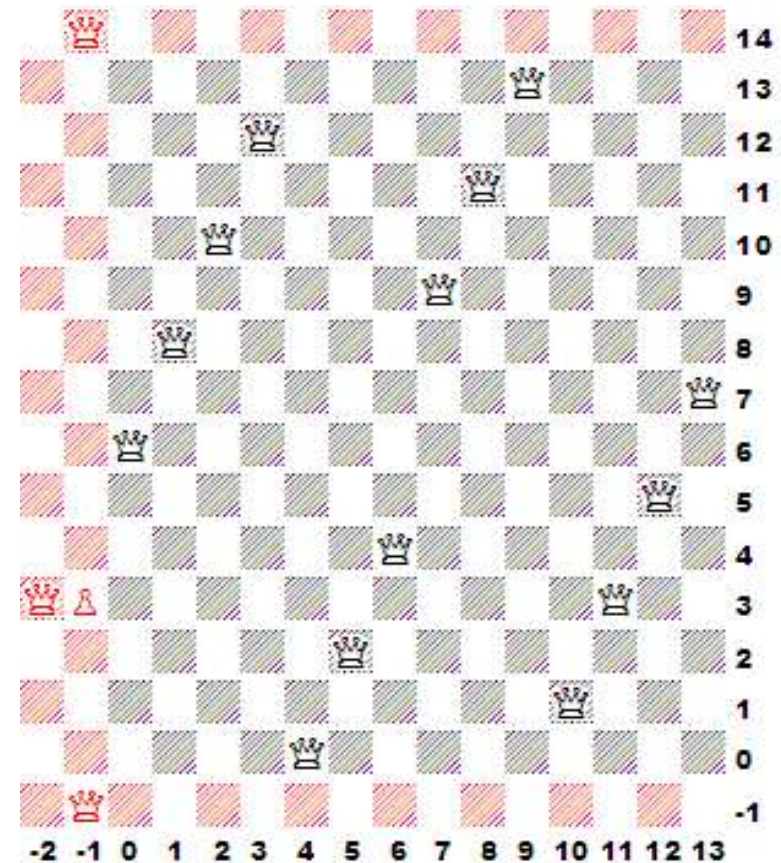
Pattern I: $N \geq 6$, $N \equiv 0$ or $2 \pmod{6}$

- Take Construction A solution to $(n-2)$ -Queens.
- Add two columns to left and one row to top and bottom
- Put pawn at $((n-2)/2 - 1, -1)$.
- Put extra queens at top and bottom of the pawn's column and to the left of the pawn.



Pattern II: $N \geq 10$, $N \equiv 0$ or $4 \pmod{6}$

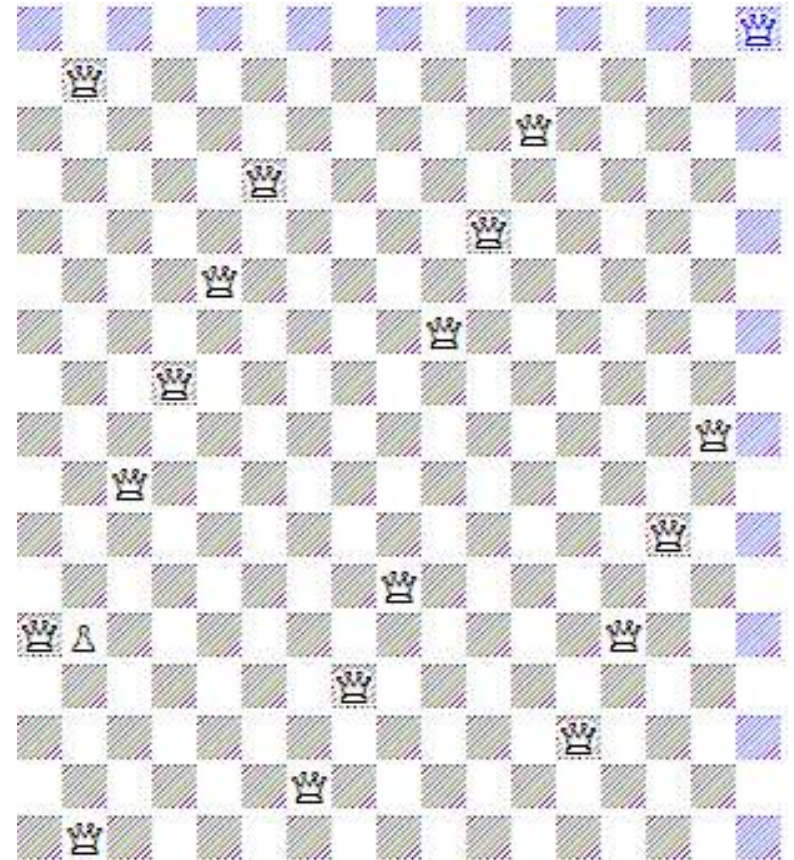
- Take Construction B solution to $(N-2)$ -Queens. Add rows and columns as in Pattern I.
- If $N=10$, $w=4$.
- If $N=12$, $w=7$.
- If $N > 12$, $w = \lfloor (N-1)/4 \rfloor$
- Pawn: $(w, -1)$
- Extra Queens: $(w, -2)$, $(-1, -1)$, $(N-2, -1)$





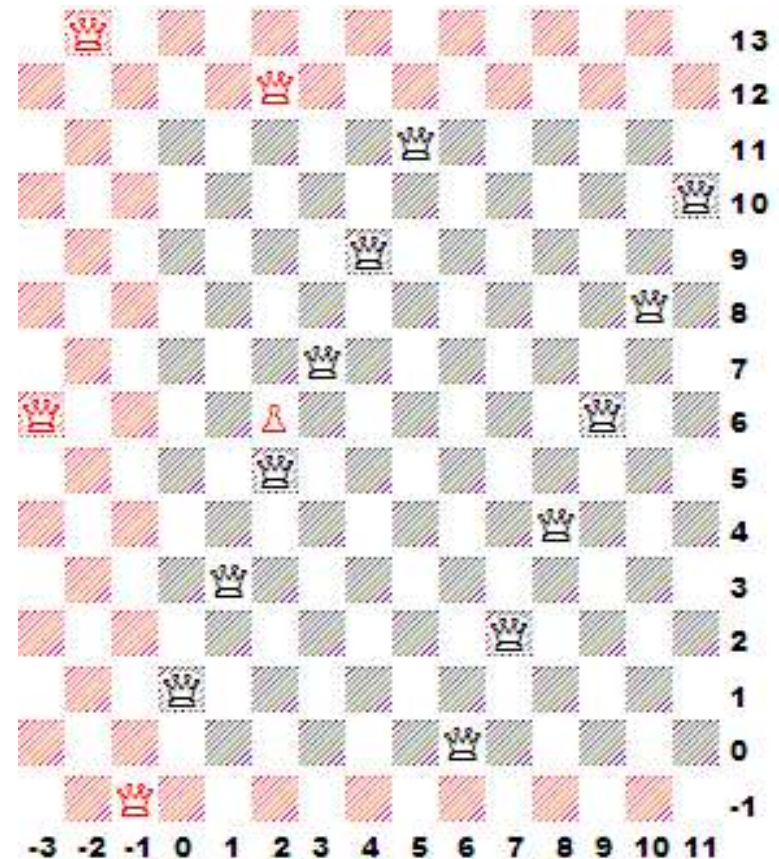
Pattern III: $N \geq 11$, $N \equiv \pm 1 \pmod{6}$

- Add row on top and column to right of Pattern II solution to $(N-1)$ -Queens.
- Pattern II leaves main diagonal open, so add a queen to the upper right corner.



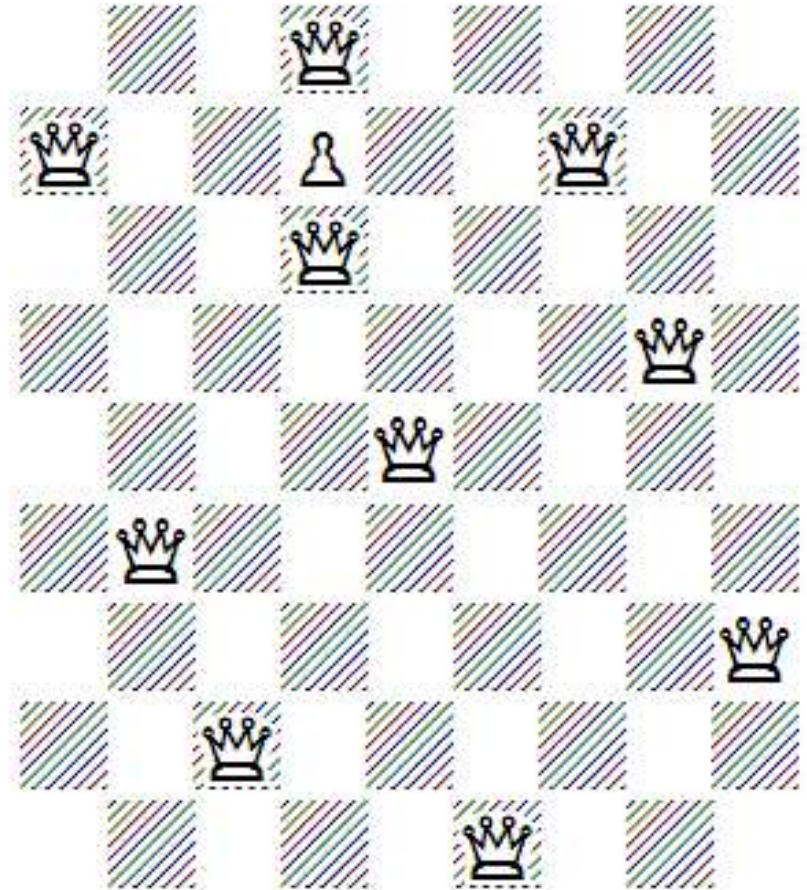
Pattern IV: $N \geq 15, N \equiv 3 \pmod{6}$

- Take Construction A solution to $(N-3)$ -Queens.
- Add 3 columns to left, 2 rows on top, 1 row on bottom.
- Pawn: $((N-3)/2, 2)$
- Extra Queens $((N-3)/2, -3), (N-2, -2), (-1, -1), (N-3, 2)$





Final Cases: $N = 7$ and $N = 9$



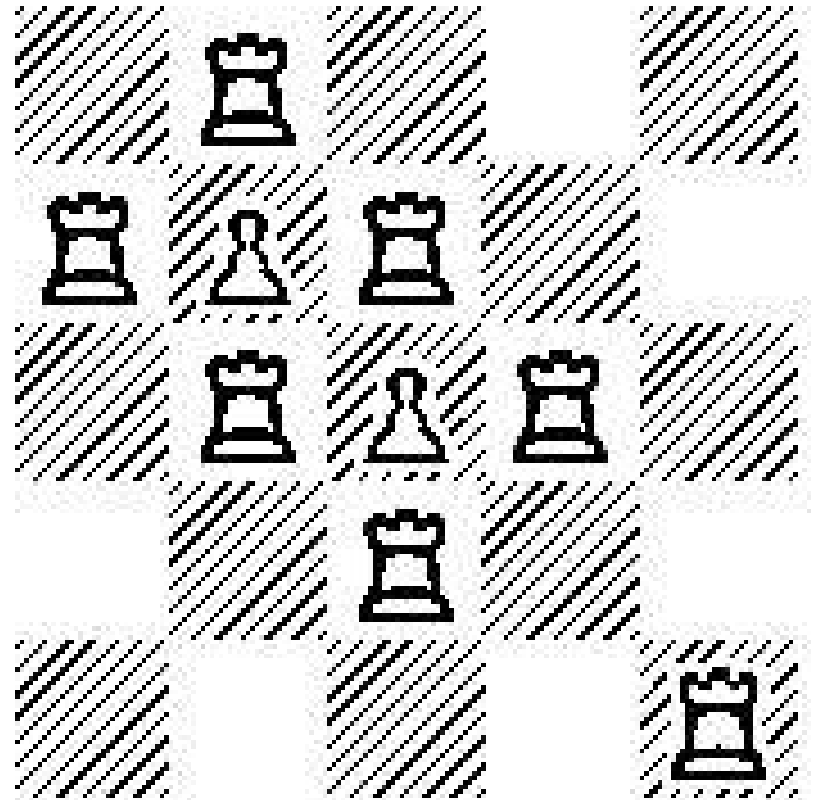


Counting $N+1$ Queens solutions

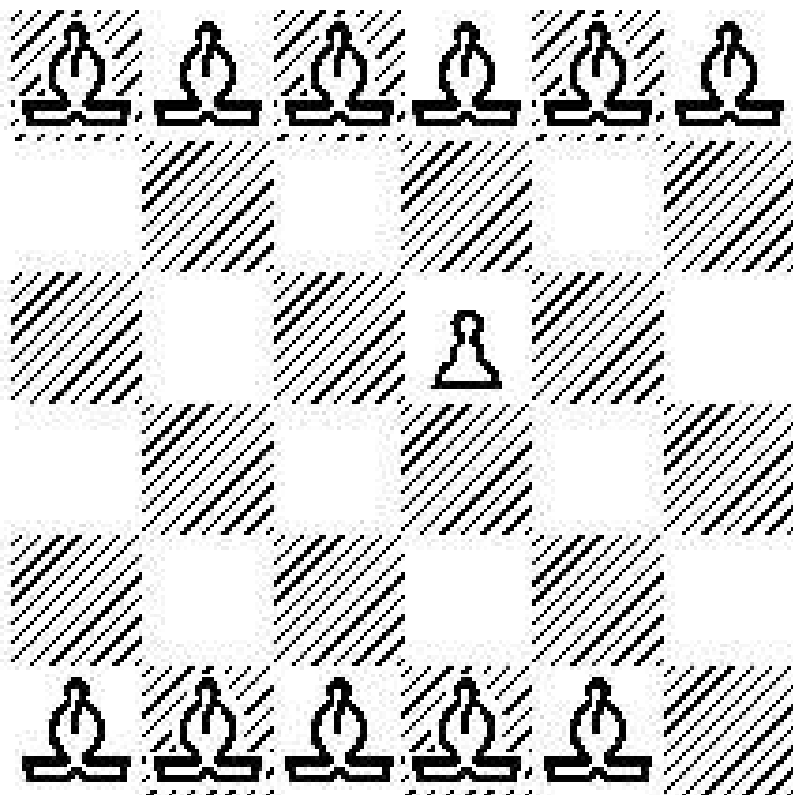
N	Number of solutions	N	Number of solutions
4	0	11	11,152
5	0	12	65,172
6	16	13	437,848
7	20	14	3,118,664
8	128	15	23,387,448
9	396		
10	2,288		

Other pieces?

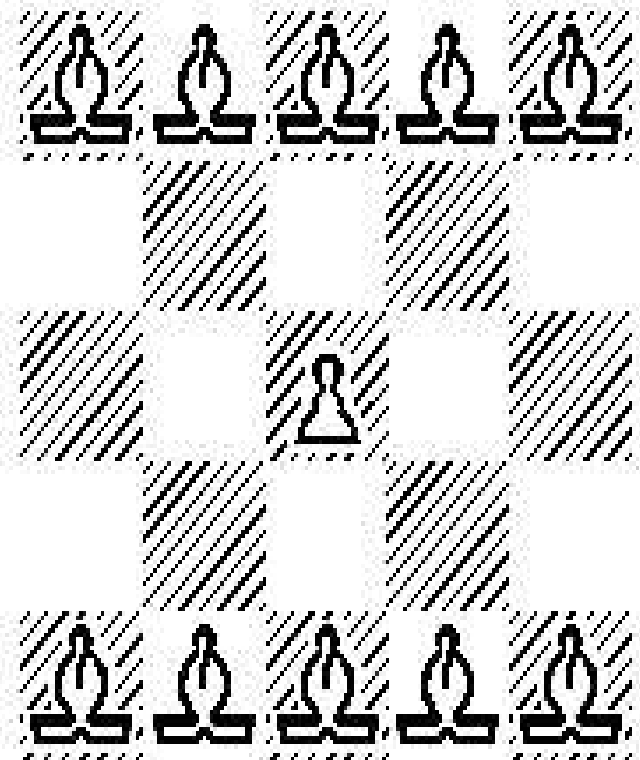
- What if we replace the queens with other pieces, such as rooks or bishops?
- For $N \geq k+2$,
 $s_R(N+k, N) = k$.



Other pieces? (Continued)



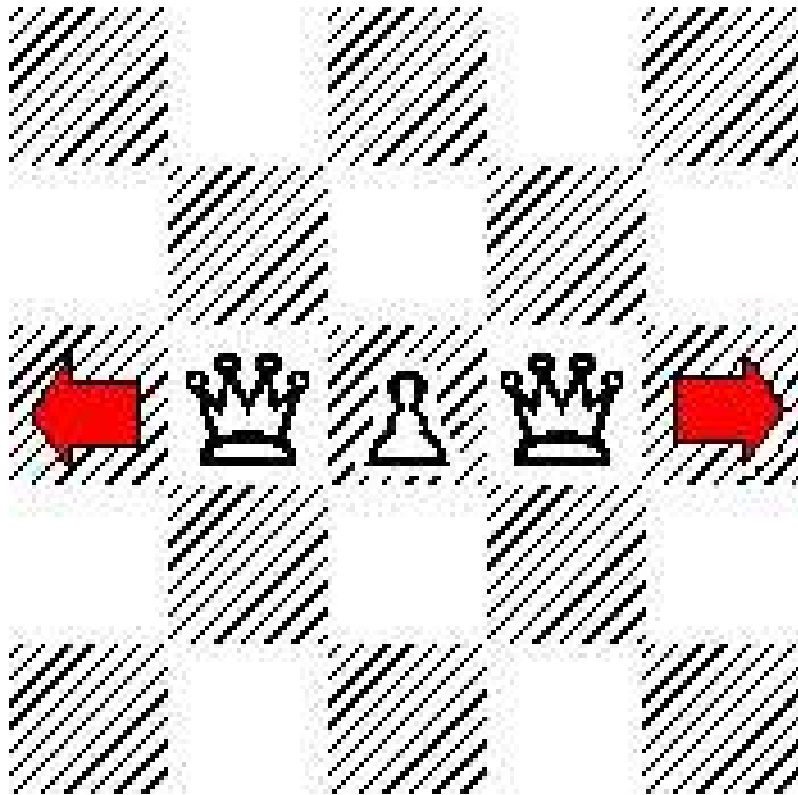
$$s_B(2n-1, n) = 1 \text{ for } n \geq 3$$



$$s_B(2n, n) = 1 \text{ for } n \geq 3 \text{ odd}$$

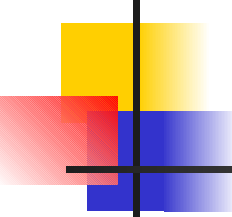


Other boards?



- What if we look at rectangular or other-shaped boards?
 - For instance, if the board is donut-shaped, one pawn won't be enough.

Conjecture:


$$s_q(N+2, N) = 2 \text{ for } N \geq 7.$$

N	Number of solutions to "N+2 Queens"	N	Number of solutions to "N+2 Queens"
5	0	10	1,304
6	0	11	12,452
7	4	12	105,012
8	44		
9	280		

Conjecture:

$$s_Q(N+3, N) = 3 \text{ for } N \geq 8.$$

N	Number of solutions to "N+3 Queens"
7	0
8	8
9	44
10	528



N+k Queens?

- Grand Conjecture: For each k , for large enough N we have $s_Q(N+k, N) = k$.
 - Is this true?
 - How much is “large enough”?
 - What happens when N isn’t “large enough”?



References

- <http://www.chessvariants.org/problems.dir/9queens.html>
 - Home of Nine Queens Contest.
- <http://www.liacs.nl/~kosters/nqueens.html>
 - Extensive bibliography on N-queens and related problems.
- Watkins, John J. (2004). *Across the Board: The Mathematics of Chess Problems*. Princeton: Princeton University Press. ISBN 0-691-11503-6.