

# Transit Graphs

Doug Chatham  
Morehead State University

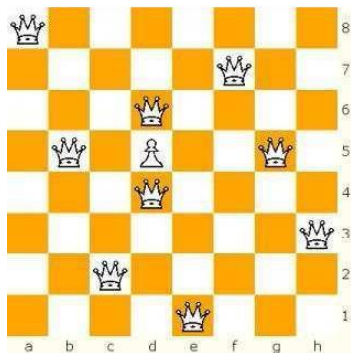
November 4, 2007

# Acknowledgments and disclaimer

- This talk represents joint work with Gerd Fricke, Joe Harless, Duane Skaggs, and Nick Wahle.
- This work was partially supported by KY NSF-EPSCoR (Grant UKRF 3046884400-07-419) and by a Morehead State University Undergraduate Research Fellowship.
- Disclaimer: All graphs are finite, simple, and loopless.

# Chessboards with pawns

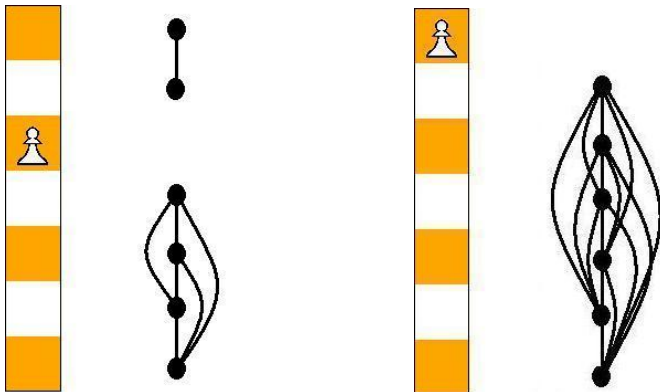
Theorem: For  $k \geq 0$ , if  $n$  is large enough, then we can place  $k$  pawns and  $n + k$  mutually nonattacking queens on an  $n \times n$  chessboard. (Chatham et. al., 2006)



<http://npluskqueens.info>

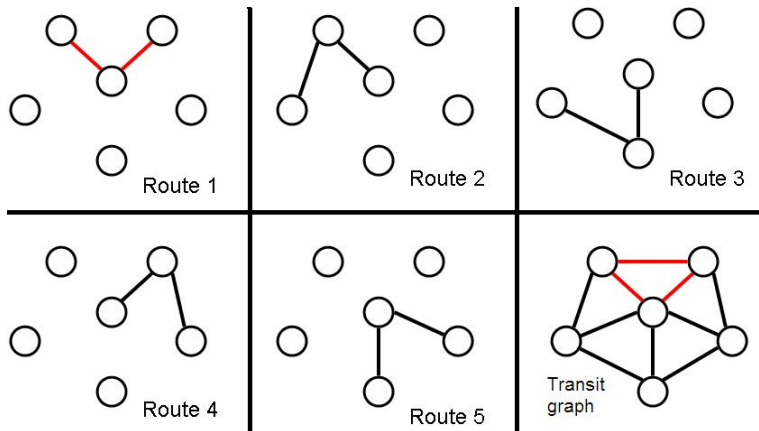
# The big question

Question: How does placing a pawn on a chessboard affect the queens graph?



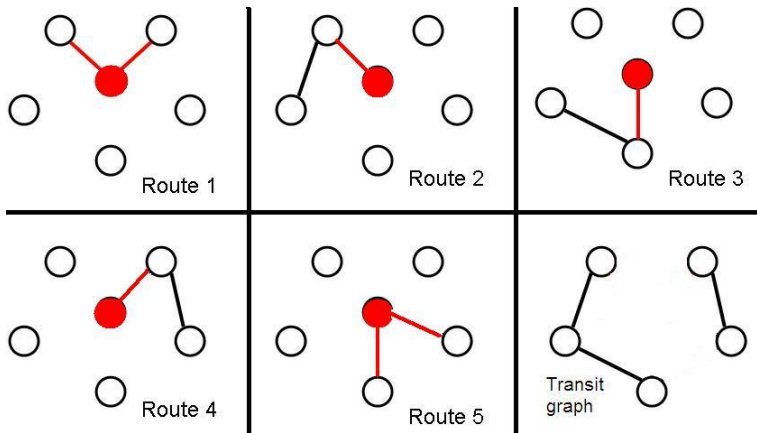
# Transit graphs

Def: Let  $F$  be a family of graphs on the same vertex set  $V$ . The *transit graph* of  $F$  is the graph on  $V$  such that  $ab$  is an edge if and only if there is a path from  $a$  to  $b$  in one of the graphs of  $F$ . The elements of  $F$  are called *routes*.



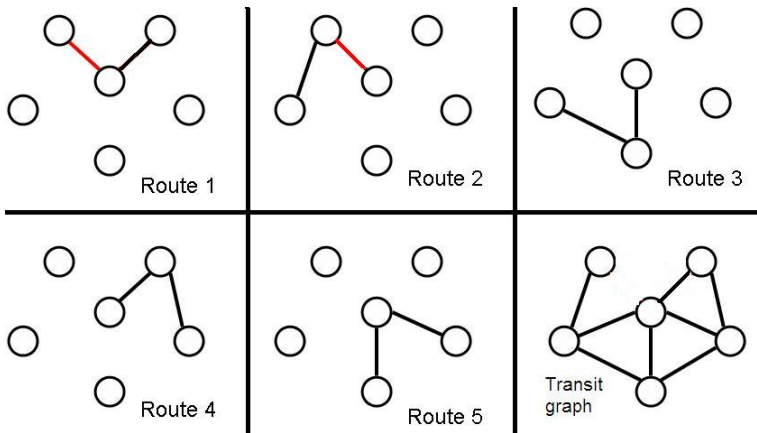
# Placing a pawn on a vertex

Def: To “place a pawn” on the vertex  $v$  on the transit graph  $G$  of some family of routes  $F$ , remove  $v$  from each member of  $F$  and create the transit graph  $G'$  of the new family of routes.

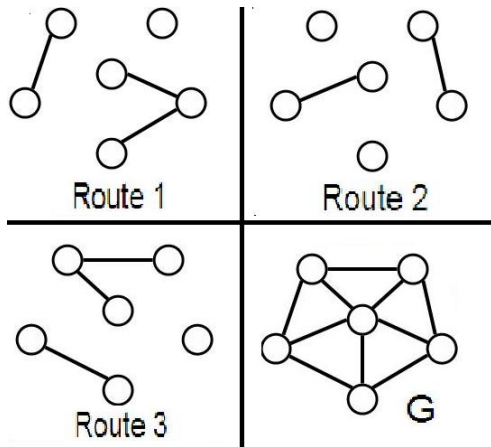


# Placing a pawn on an edge

Def: To “place a pawn” on an edge  $ab$  on the transit graph  $G$  of some family of routes  $F$ , remove  $ab$  from each member of  $F$  and create the transit graph  $G'$  of the new family of routes.



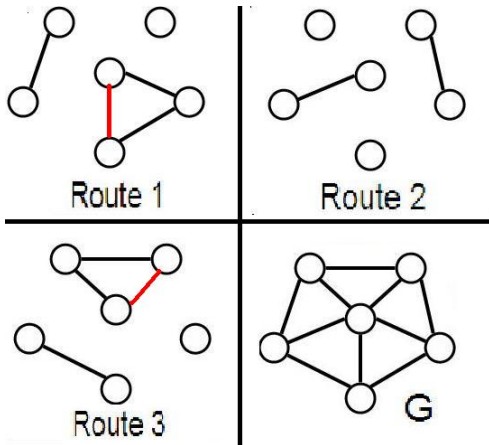
# Minimizing the number of routes



Def: The *equivalence number* of  $G$ ,  $eq(G)$ , is the minimum cardinality of a family of routes for which  $G$  is the transit graph.

# Why call it an equivalence number?

Take each route and complete its connected components. The number of routes is unchanged.



Each route represents an equivalence relation. See (Duchet, 1978) and (Frankl, 1982).

# Edge subchromatic number

- If we replace “cover” with “partition”, we get the *edge subchromatic number*  $\chi'_s(G)$  (Domke, Laskar, Hedetniemi, 1988).
- Also known as  $\bar{e}(G)$  (Duchet, 1978).
- Observation (Duchet, 1978): For any graph  $G$ ,  $eq(G) \leq \chi'_s(G)$ . If  $G$  has no triangles, then  $\chi(G) = \chi'_s(G) = eq(G)$ .

# More on equivalence and edge subchromatic numbers

- $eq(P_n) = 2$
- $eq(C_n) = 2$  if  $n$  is even and 3 if  $n$  is odd.
- For any tree  $T$ ,  $eq(T) = \Delta(T)$
- Proposition (Duchet, 1978):  $\chi'_s(G) - eq(G)$  can be arbitrarily large.

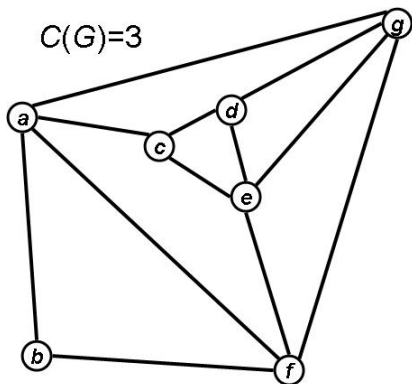
*Proof sketch: Consider  $K_n$  with an edge  $ab$  removed.*

*Now,  $eq(K_n \setminus ab) = 2$  but  $\chi'_s(K_n \setminus ab) \geq 2\lceil\sqrt{n-2}\rceil$ .*

- Proposition:  $eq(G) + eq(G^c) \leq 2n - 1$  and  $eq(G)eq(G^c) \leq n^2 - n$ .

# Wahle's $c(v)$

Def: Let  $G = (V, E)$  be a graph. For each  $v \in V$ , we define  $c(v)$  to be the minimum number of cliques needed to cover the edges of  $G$  incident with  $v$ . We define  $c(G) = \max_{v \in V} c(v)$ .

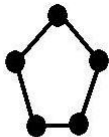


$v$	$c(v)$
$a$	3
$b$	1
$c$	2
$d$	2
$e$	2
$f$	2
$g$	2

# More on $c(G)$ and $eq(G)$

Proposition: For any graph  $G$ ,  $c(G) \leq eq(G)$ .

Conjecture: For any graph  $G$ ,  $eq(G) \leq c(G) + 1$ .



$$c(G) = 2$$
$$eq(G) = 3$$

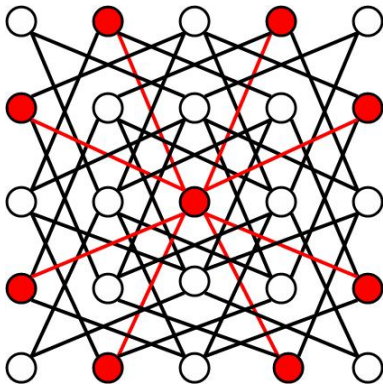
Observation: If  $eq(G) = k$ , then  $G$  has no induced subgraph isomorphic to  $K_{1,k+1}$ . The converse is false.

(See Ryjáček and Schiermeyer, 1995 for bounds on independence numbers of such graphs.)

# Equivalence numbers for some chessboard graphs

Proposition:

- 1  $eq(R_n) = 2$  for  $n \geq 2$ .
- 2  $eq(Q_n) = 4$  for  $n \geq 4$ .
- 3  $eq(B_n) = 2$  for  $n \geq 2$ .
- 4  $eq(N_n) = 8$  for  $n \geq 5$ .



- R. D. Chatham, M. Doyle, G. H. Fricke, J. Reitmann, R. D. Skaggs, and M. Wolff, *Independence and domination separation in chessboard graphs*, J. Combinat. Math. and Combinat. Comp., to appear.
- G. S. Domke, R. Laskar, and S. T. Hedetniemi, *The edge subchromatic number of a graph*, Congr. Numer. 64 (1988), 95-104.
- P. Duchet, *Colorations transitives des graphes*, Cahiers Centre Études Rech. Opér. 20 (1978), no. 3-4, 363-371.
- P. Frankl, *Covering graphs by equivalence relations*, Theory and practice of combinatorics, 125-127, North-Holland Math. Stud., 60, North-Holland, Amsterdam, 1982.
- Z. Ryjáček and I. Schiermeyer, *On the independence number in  $K_{1,r+1}$ -free graphs*, Discrete Math. 138 (1995), no. 1-3, 365-374.