N-dimensional Pairs (Preliminary Report)

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Introduction

Note: All rings in this talk are commutative with $1 \neq 0$ and all subrings are unital.

$\dim R$: Krull dimension of R

Def: Let n be a nonnegative integer or ∞. If $R \subseteq T$ are rings such that $\dim A = n$ for every ring A for which $R \subseteq A \subseteq T$, then we say that (R, T) is an n-**dimensional pair**.

Example:

If $R \subseteq T$ is an integral extension, then (R, T) is an n-dimensional pair, where $n := \dim R$.

$Spec(R)$: the set of prime ideals of R

$Min(R)$: the set of minimal prime ideals of R

LO: lying-over property

INC: incomparability property

GU: going-up property

Def: Let $R \subseteq T$ be a ring extension. We say (R, T) is an LO-pair (resp. INC-pair) if $A \subseteq B$ satisfies LO (resp., INC) for all rings A and B such that $R \subseteq A \subseteq B \subseteq T$.

Folklore Theorem:
The ring extension $R \subseteq T$ is integral $\Longleftrightarrow$ (R, T) is both an INC-pair and an LO-pair.

(Dobbs 1981) If (R, T) is an LO-pair, then $\dim R \leq \dim A \leq \dim R + 1$ for every ring A such that $R \subseteq A \subseteq T$.

(Dobbs 1981) If T is quasisemilocal, then (R, T) is an LO-pair $\iff$ T is integral over R.

Example:

If (R, T) is an LO-pair of rings with $\dim R = \infty$, then we have that (R, T) is an ∞-dimensional pair.

Example:

Let F be a field and $F(X) + M$ be an ∞-dimensional valuation domain with maximal ideal M. Then we have that $(F + M, F[X] + M)$ is an ∞-dimensional LO-pair which is not integral.

$t.d._R(T)$: transcendence degree of $qf(T)$ (the quotient field of T) over $qf(R)$

Def: If $R \subseteq T$ are domains such that $qf(R) = qf(T)$ then T is an *overring* of R.

$\dim_v R$: valuative dimension of $R = \sup\{\dim V \mid V$ is a valuation overring of $R\}$ (when R is a domain)

Def: A finite-dimensional domain R is *Jaffard* if $\dim R = \dim_v R$.

Def: A ring R is *residually Jaffard* if R/P is Jaffard for all $P \in Spec(R)$.

$Nil(R)$: the ideal of nilpotent elements of R

Def: $R_{red} = R/Nil(R)$

Results

Proposition 1 *Let (R, T) be a LO-pair of rings with R residually Jaffard and $n := \dim R < \infty$. Then (R, T) is an n-dimensional pair $\iff t.d._{R/(P \cap R)} T/P = 0$ for all $P \in Min(T)$.*

Proposition 2 *Let R be a Jaffard domain with $\dim R = n$ and (R, T) an LO-pair of domains. Then we have that (R, T) is an n-dimensional pair $\iff$ $R \subseteq T$ is an algebraic extension.*

Example:

(Debremaeker and Van Lierde, 2000) *Let* K *be a field,* $R = K[X, XY, XY^2, \ldots]$*, and* $T = K[X, Y]$*. Then* (R, T) *is a* 2*-dimensional* LO*-pair that is not an* INC*-pair.*

In the above example, $A \subseteq T$ *is integral for every ring* A *for which* $R \subsetneq A \subseteq T$*.*

Proposition 3 *Let* (R, T) *be a* 1*-dimensional* LO*-pair of rings with* R *residually Jaffard. Then* $R \subseteq T$ *is an integral extension.*

Proposition 4 *Let* R *be a* 1*-dimensional Jaffard domain and* T *an overring of* R *strictly contained in* $qf(R)$*. Then* (R, T) *is a* 1*-dimensional pair.*

In the above Proposition, (R, T) *need not be a* LO*-pair. For example, take* $R = \mathbb{Z}$ *and* $T = \mathbb{Z}_{p\mathbb{Z}}$*.*

Proposition 5 *Let R be a maximal non-Jaffard subring of its quotient field K. Let V be a valuation overring of R with* $\dim V = \dim_v R$ *and let T be an subring of V strictly containing R. Then (T, V) is an n-dimensional pair with $n =$* $\dim T$.

Proposition 6 *Let (R, T) be an n-dimensional pair of domains with $n < \infty$. Then (R, T) is a Jaffard domain pair $\iff$ R is Jaffard.*

Example: *(Oukessou and Miri, 1999) Let a domain T be a minimal overring of a nonfield R such that $(R:T) = 0$. Then $\dim R = \dim T$. Additionally, T is integral over $R \iff R \subseteq T$ satisfies LO.*

Proposition 7 *Let (R,T) be an INC-pair of domains with R quasilocal and integrally closed in T. Let $\dim R < \infty$ and $R \neq T$. Then (R,T) is not an n-dimensional pair.*

Proposition 8 *Let $R \subseteq T$ be a ring extension. Then*

1. (R, T) *is an n-dimensional pair* $\iff$ (R_{red}, T_{red}) *is an n-dimensional pair.*

2. $R \subseteq T$ *is integral* $\iff$ $R_{red} \subseteq T_{red}$ *is integral.*

Thus, when looking for non-integral n-dimensional pairs, we can restrict our attention to reduced rings.

Example: *(Ben Nasr, 2002) Let S be a two-dimensional Prüfer domain and M a height one maximal ideal of S. Consider D a subfield of the residue field $F = S/M$ such that $t.d._D F = 1$ (for instance $S = V_1 \cap V_2$ where $V_1 = K(X)[Y]_{(Y)}, V_2 = K(Y)+XK(Y)[X]_{(X)}$, and $D = K$). Let $R = (S, M, D)$. Then we have that (R, S) is a 2-dimensional pair.*

We can show that the example above is neither an INC-pair nor an LO-pair.

We can generate higher-dimensional examples with the following:

Proposition 9 *Let V be a valuation domain of the form $F + M$, where F is a field and M is the maximal ideal of V. Let $R \subseteq T$ be subrings of F. Let $n = \dim R$ and $m = \dim V$. Then $(R{+}M, T{+}M)$ is an $(n{+}m)$-dimensional pair $\iff$ (R, T) is an n-dimensional pair. (Correction made during meeting: We must also assume $m < \infty$.)*