Going-Down Pairs of Commutative Rings

Doug Chatham

March 16, 2002

Background

GD: going-down property (of ring extensions)
INC: incomparability property
LO: lying-over property
GU: going-up property
$Spec(R)$: prime ideals of R with the Zariski topology
$Min(R)$: minimal prime ideals of R
dim(R): Krull dimension of R
$t.d._R(T)$: transcendence degree of the quotient field of T over the quotient field of R

Def: A ring extension $R \subseteq T$ is **open** if the contraction map $Spec(T) \to Spec(R)$ given by $P \mapsto P \cap R$ is an open topological map.

Def: Let $\mathcal{P}$ be a property of ring extensions and let $R \subseteq T$ be a ring extension. Then (R, T) is a $\mathcal{P}$-**pair** if $A \subseteq B$ satisfies $\mathcal{P}$ for all ring extensions $A \subseteq B$ such that $R \subseteq A \subseteq B \subseteq T$.

LO-pairs, GU-pairs, and INC-pairs have been studied previously in the literature.

Def: Let $\mathcal{P}$ be a property of ring extensions. A domain R is a $\mathcal{P}$-**domain** if for all overrings S of R, $R \subseteq S$ satisfies $\mathcal{P}$.

Ex: If a domain R is Prüfer or has dim$(R) \leq 1$, then R is a GD-domain.

Ex: Let R be a Noetherian domain. Then R and all its overrings are open-domains $\iff$ R is semilocal and dim$(R) \leq 1$ (Papick, 1976).

Def: We say (R, T) is a $\mathcal{P}$-**domain pair** if S is a $\mathcal{P}$-domain, for all S such that $R \subseteq S \subseteq T$.

Ex: If R is a Prüfer domain and K is its quotient field, then (R, K) is a GD-domain pair.

Ex: If $k \subseteq K$ is an algebraic field extension, then (k, K) is an open-domain pair.

Ex: Let R be Noetherian and k the quotient field of R. Then (R, k) is an open domain pair $\iff$ R is semilocal and $\dim(R) \leq 1$. (Restatement of above example of Papick)

Def: A ring R is a $\mathcal{P}$-**ring** if $\forall P \in Spec(R)$, R/P is a $\mathcal{P}$-domain. (For $\mathcal{P} = GD$, Dobbs 1996)

Ex: dim$(R) \leq 1 \Rightarrow R$ is a GD-ring. Other examples of GD-rings include pseudo-valuation rings, chained rings, and finite products of GD-rings.

If R is a domain, then R is a GD-ring (resp., open-ring) $\iff$ R is a GD-domain (open domain).

Def: We say (R, T) is a $\mathcal{P}$**-ring pair** if S is a $\mathcal{P}$-ring, $\forall S \ni R \subseteq S \subseteq T$.

Proposition 1 *The following are equivalent:*

1. *(R, T) is a GD-ring pair (resp., open-ring pair);*

2. *For each $Q \in Spec(T)$, $(R/(Q \cap R), T/Q)$ is a GD-domain pair (open domain pair);*

3. *For each $Q \in Min(T)$, $(R/(Q \cap R), T/Q)$ is a GD-domain pair (open domain pair).*

Transcendence degree results

Lemma 2 *If k is a field contained in a domain T, then (k, T) is a GD-domain pair* $\iff$ $t.d._k(T) \leq 1$.

Theorem 3 *If* $\dim(R) = 0$, *then the following are equivalent:*

1. (R, T) *is a GD-ring pair;*
2. $t.d._{R/P\cap R}(T/P) \leq 1, \forall P \in Spec(T)$;
3. $t.d._{R/P\cap R}(T/P) \leq 1, \forall P \in Min(T)$.

Lemma 4 *If k is a field contained in a domain T, then (k, T) is an open-ring pair $\iff k \subseteq T$ is an algebraic field extension.*

Theorem 5 *If* $\dim(R) = 0$, *then the following are equivalent:*

1. (R, T) *is a open-ring pair;*

2. $t.d._{R/P\cap R}(T/P) = 0, \forall P \in Spec(T)$;

3. $t.d._{R/P\cap R}(T/P) = 0, \forall P \in Min(T)$;

4. (R, T) *is an* INC*-pair;*

5. T *is an integral extension of* R.

What if $\dim(R) \geq 1$?

Proposition 6 *$R[X]$ is a GD-ring $\iff$ $\dim(R) = 0$.*

Corollary 7 *If (R, T) is a GD-ring pair and $\dim(R) > 0$, then $R \subseteq T$ is an algebraic ring extension.*

The converse of this corollary fails: consider $(\mathbb{Z}[X], \mathbb{Z}[X][\sqrt{2}])$.

GD-ring pairs need not be INC-pairs, even when $\dim(R) > 0$.

Proposition 8 *If $\dim(R) \leq 1$ and (R, T) is an INC-pair, then (R, T) is a GD-ring pair.*

For $\dim(R) \geq 2$, INC-pairs need not be GD-ring pairs.

Proposition 9 *If* (R, T) *is an open-ring pair, then* (R, T) *is an* INC*-pair.*

Proposition 10 *Let* (R, T) *be a normal pair of domains (i.e.* A *is integrally closed in* $B, \forall R \subseteq A \subseteq B \subseteq T$*). Then* (R, T) *is a* GD*-domain pair (resp., open domain pair)* $\iff$ R *is a* GD*-domain (open domain).*

Relation between $\mathcal{P}$-ring pairs and $\mathcal{P}$-pairs

If R is a GD-domain, $R \subseteq T$ satisfies GD for all domains T containing R. (Dobbs-Papick, 1976)

So (R, T) GD-domain pair $\Rightarrow$ (R, T) GD-pair.

If R is a open domain, $R \subseteq T$ is open for all domains T containing R. (Papick, 1976)

So (R, T) open domain pair $\Rightarrow$ (R, T) open pair.

Proposition 11 $(\mathbb{F}_p, \mathbb{F}_p(X))$ *is an* INC*-domain pair that is not an* INC*-pair.*

Also, R a GD-ring does **not** imply $R \subseteq T$ satisfies GD for all rings T containing R. (Dobbs, 1996)

Theorem 12 *Let* $\mathcal{P} \in \{GD, open\}$. *Then* R *a* $\mathcal{P}$*-ring* $\Rightarrow$ $R \subseteq T$ *satisfies* $\mathcal{P}$ *for all* T *with a unique minimal prime.*

Corollary 13 *Let* $\mathcal{P} \in \{GD, open\}$. *If* T *is a ring with a unique minimal prime and* (R, T) *is a* $\mathcal{P}$*-ring pair, then* (R, T) *is a* $\mathcal{P}$*-pair.*

Are $\mathcal{P}$-pairs always $\mathcal{P}$-ring pairs?

Ex: Let $k \subseteq K$ be an algebraic field extension and T a ring that is *not* a GD-ring. Then $(k \times T, K \times T)$ is a GD-pair and an open-pair that is not a GD-ring pair (or an open-ring pair).

Ex: Let $L \subset K$ be an algebraic extension of distinct fields. Let $T = K[[X,Y]]$ and $R = L + XT + YT$. Then (R,T) is a GD-pair of domains that is not a GD-domain pair.

Flat pairs and flat-domain pairs

Def: A ring extension $R \subseteq T$ is **flat** if T is a flat R-module.

A domain R is a flat-domain $\iff$ R is a Prüfer domain (Richman, 1965)

Therefore, we will refer to a flat-domain pair as a "Prüfer domain pair."

Proposition 14 *Let R be a domain contained in a field K. Then the following are equivalent:*

1. *(R, K) is a flat pair;*

2. *(R, K) is a Prüfer domain pair;*

3. **either**

   (a) $R \subseteq K$ is an algebraic field extension;

   or

   (b) R is a Prüfer domain and K is the quotient field of R.

Proposition 15 *Let $R \subseteq T$ be domains with R Prüfer. Then the following are equivalent:*

1. (R, T) *is a flat pair;*

2. (R, T) *is a Prüfer domain pair;*

3. **either**

   (a) $R \subseteq T$ *is an algebraic field extension;*

   or

   (b) T *is an overring of* R.