Houston Journal of Mathematics

Volume 33, No. 3, 2007

PAIRS OF COMMUTATIVE RINGS IN WHICH ALL INTERMEDIATE RINGS HAVE THE SAME DIMENSION

R. DOUGLAS CHATHAM AND DAVID E. DOBBS

Communicated by Johnny A. Johnson

Abstract. If $0 \leq n \leq \infty$ and $R \subseteq T$ are (commutative unital) rings, then (R, T) is called an n-dimensional pair if each intermediate ring A (that is, each ring A such that $R \subseteq A \subseteq T$) has Krull dimension n. If $1 \leq n \leq \infty$, examples are given of n-dimensional pairs that are not integral extensions, including an infinite family of n-dimensional pairs that are neither LO-pairs nor INC-pairs. The n-dimensional pair property transfers well in constructions involving pullbacks or passage to the associated reduced rings, but this property is not stable under passage to factor domains. Special attention is paid to the n-dimensional pairs whose first coordinate is a Jaffard domain or a residually Jaffard ring. Also, examples are given of ∞-dimensional pairs whose intermediate rings have prime ideal chains of arbitrarily large cardinality; and of a family of n-dimensional pairs arising from minimal overrings.

1. Introduction

All rings considered in this note are commutative with $1 \neq 0$, and all subrings are unital. If A is a ring, the set of prime (resp., minimal; resp., maximal) prime ideals of A is denoted by $\mathrm{Spec}(A)$ (resp., $\mathrm{Min}(A)$; resp., $\mathrm{Max}(A)$); and the (Krull) dimension of A is denoted by $\dim(A)$. If $\mathcal{P}$ is a ring-theoretic property and $R \subseteq T$ are rings, then (R, T) is called a $\mathcal{P}$-*pair* if A satisfies $\mathcal{P}$ for each intermediate ring A (that is, for each ring A such that $R \subseteq A \subseteq T$). Special attention will be paid to the LO-pairs and the INC-pairs. (Following [16, page 28], we let LO, INC

2000 *Mathematics Subject Classification.* Primary 13B02, 13A15, 13C15. Secondary 13G05, 13B21.

Key words and phrases. n-dimensional pair, LO-pair, INC-pair, integral extension, Jaffard domain, residually Jaffard ring, overring.

and GU denote the lying-over, incomparability and going-up properties, respectively, of ring extensions.) It is known (see [7, page 70], [1, Theorem 2.3] and [13, Theorem 6.5.6]) that for a ring extension $R \subseteq T$, we have that (R, T) is an INC-pair if and only if (R, T) is a residually algebraic pair (i.e., if and only if $R/(P \cap R) \subseteq A/P$ is algebraic for every prime ideal P of every intermediate ring A). For more background on INC-pairs, the reader is directed to [1] and [13]; for more background on LO-pairs, see [9]. These types of pairs are significant, in part, because of the Folklore Theorem [9, page 454]: a ring extension $R \subseteq T$ is integral if and only if (R, T) is both an INC-pair and an LO-pair. Since the partners of any integral extension have the same dimension (cf. [16, Theorem 48]), any integral extension $R \subseteq T$ gives rise to an n-dimensional pair, in the sense of the following definition. If $0 \leq n \leq \infty$ and $R \subseteq T$ are rings, then (R, T) is called an *n-dimensional pair* if $\dim(A) = n$ for all rings A such that $R \subseteq A \subseteq T$ (i.e., if $\dim(A) = \dim(R)$ for all intermediate rings A, with $n := \dim(R)$). Much of our focus will be on producing examples of n-dimensional pairs that fail to have one or both of the LO-pair and INC-pair properties. There is no need to consider further the case $n = 0$, as it follows easily from the Folklore Theorem that any 0-dimensional pair is integral (cf. [16, Exercise 1, page 41]). Also, since it follows from the Krull-Akizuki Theorem that (R, T) is a 1-dimensional pair whenever T is an overring of a one-dimensional Noetherian integral domain R other than the quotient field of R, we pay special attention to the n-dimensional pairs whose first coordinate is a Jaffard domain or a residually Jaffard ring.

Before giving a summary of our results, we devote this paragraph to additional background. Let R be a(n integral) domain. We let $\mathrm{qf}(R)$ denote the quotient field of R. As in [14], an *overring* of R is any ring T such that $R \subseteq T \subseteq \mathrm{qf}(R)$. The *valuative dimension* of R is defined as $\dim_v(R) := \sup\{\dim(V) \mid V$ is a valuation overring of $R\}$. In general, $\dim(R) \leq \dim_v(R)$. A finite-dimensional domain R is called a *Jaffard domain* if $\dim(R) = \dim_v(R)$; and a ring R is called *residually Jaffard* if R/P is a Jaffard domain for all $P \in \mathrm{Spec}(R)$. Each residually Jaffard domain is (obviously) a Jaffard domain, but the converse is false. Any finite-dimensional Noetherian ring is a residually Jaffard ring.

Since $\dim(A) \geq \dim(R)$ for any intermediate ring A of any LO-pair (R, T) [9, Corollary 3.11], it follows that if T is an overring of a Jaffard domain R such that (R, T) is an LO-pair, then (R, T) is an n-dimensional pair. A generalization of this fact is given in Proposition 2.1 which, for Jaffard domains R, characterizes the n-dimensional pairs of domains (R, T) that are also LO-pairs. Proposition 2.2 does the analogue for the finite-dimensional residually Jaffard rings R, via transcendence degree conditions in the spirit of [5]. In the case of one-dimensional such rings R, Proposition 2.5 shows that these pairs arise only from integral extensions (and, hence, must be INC-pairs). However, in general, an n-dimensional pair that is an LO-pair (resp., an INC-pair) need not be an INC-pair (resp., an LO-pair): see Examples 2.3 and 2.4 (resp., the comment following the proof of Proposition 2.6). Moreover, there exist n-dimensional pairs which are *neither* LO-pairs nor INC-pairs. For $n = 2$, see Example 2.8 for such an example, which features an analysis of a particular Jaffard domain pair constructed by Ben Nasr [2]. For $3 \leq n < \infty$, one then applies a transfer result involving pullbacks (see Example 2.9 and the second paragraph following its proof). While the class of n-dimensional pairs behaves similarly to the classes of LO-pairs, of INC-pairs, and of integral pairs with respect to transfer in constructions arising from pullbacks and in the passage to the associated reduced rings (see Proposition 2.10), further analysis of the construction of Ben Nasr shows that, unlike the more familiar classes, the class of n-dimensional pairs of domains is not stable under formation of factor domains. For additional results on n-dimensional pairs concerning Jaffard domains or maximal non-Jaffard subrings of a field (in the sense of [3]), see Propositions 2.11 and 2.16.

Many of the results in this paper are restricted to pairs of finite-dimensional rings. However, the diverse nature of ∞-dimensional pairs is apparent from several examples which can be found in [15, Example 3.2, Theorem 4.4, Corollary 4.5, and Corollary 4.7]; in fact, these give instances of rings R for which (R, T) is an ∞-dimensional pair for any extension ring T of R. Proposition 2.13 and Examples 2.14 and 2.15 present new examples of ∞-dimensional pairs; those in Examples 2.14 and 2.15 are non-integral pairs featuring arbitrarily large chains of prime ideals in all the intermediate rings. The final result of the paper indicates how a certain type of minimal overring (in the sense of [11], [18]) leads to a non-integral n-dimensional pair.

In addition to the above notation and conventions, we let $\subset$ denote proper inclusion; and for domains $A \subseteq B$, we let $\text{t.d.}_A(B)$ denote the transcendence degree of $\text{qf}(B)$ over $\text{qf}(A)$. If A is a ring, then $\text{Nil}(A)$ denotes the nilradical of A; and $A_{red} := A/\text{Nil}(A)$, the associated reduced ring of A. Any unexplained material is standard, as in [13], [14], [16].

2. Results

We begin with some transcendence degree criteria for n-dimensional pairs.

Proposition 2.1. *Let $R \subseteq T$ be domains such that R is a Jaffard domain and (R,T) is an LO-pair. Then (R,T) is an n-dimensional pair $\iff$ $R \subseteq T$ is an algebraic extension.*

Proof. Let $n := \dim(R)$. Since (R,T) is an LO-pair, [9, Corollary 3.1] ensures that $\dim(A) \geq n$ for all rings A such that $R \subseteq A \subseteq T$. Thus, (R,T) is an n-dimensional pair if and only if $\dim(A) \leq n$ for all such A. It now suffices to observe, by [6, Proposition 2.2], that $\dim(A) \leq n$ for all intermediate rings A if and only if $\text{t.d.}_R(T) = 0$, i.e., if and only if T is algebraic over R. □

We next give an analogue of the above result for rings with zero-divisors.

Proposition 2.2. *Let (R,T) be an LO-pair of rings such that R is residually Jaffard and $n := \dim(R) < \infty$. Then (R,T) is an n-dimensional pair $\iff$ $t.d._{R/(P\cap R)}(T/P) = 0$ for all $P \in Min(T)$.*

Proof. As in the preceding proof, an appeal to [9, Corollary 3.11] yields that (R,T) is an n-dimensional pair if and only if $\dim(A) \leq n$ for all rings A such that $R \subseteq A \subseteq T$. The proof is completed by observing, via [6, Proposition 2.5], that $\dim(A) \leq n$ for all such A if and only if $\text{t.d.}_{R/(P\cap R)}(T/P) = 0$ for all $P \in \text{Min}(T)$. □

The next two results summarize some known examples which can be interpreted to imply that an n-dimensional pair need not be an INC-pair.

Example 2.3. *Let F be a field and $F(X) + M$ be an ∞-dimensional valuation domain with maximal ideal M. Then $(F + M, F[X] + M)$ is an ∞-dimensional LO-pair which is not integral (i.e., which is not an INC-pair).*

Proof. Nearly everything is accomplished by [9, Remark 3.12 (a)]. It remains only to verify the following part of the assertion: if B is a ring such that $F+M \subset B \subset F[X] + M$, then $\dim(B) = \infty$. As $\dim(F[X] + M) = \infty$, it suffices to show that $F[X] + M$ is integral over B. Hence, by [9, Theorem 4.1], we need only prove B is not a field, and this, in turn, holds since M is a nonzero proper ideal of B. □

Example 2.4. *(Debremaeker and Van Lierde [8]) Let K be a field, let X and Y be algebraically independent over K, and set $R := K[X, XY, XY^2, \ldots, XY^n, \ldots]$. Then (R,T) is a 2-dimensional LO-pair which is not integral (i.e., which is not an INC-pair).*

The (at least) two-dimensionality of the preceding example is unavoidable, as we next obtain different behavior in the one-dimensional case.

Proposition 2.5. *Let (R,T) be a 1-dimensional LO-pair of rings with R residually Jaffard. Then $R \subseteq T$ is an integral extension (and, hence, both an LO-pair and an INC-pair).*

Proof. By Proposition 2.2, $\text{t.d.}_{R/P\cap R}(T/P) = 0$ for all minimal primes P of T. On the other hand, since (R,T) is an LO-pair, [9, Corollary 3.9(a)] yields that $\text{t.d.}_{R/P\cap R}(T/P) = 0$ for all maximal primes P of T. Since $\dim(T) = 1$, we conclude that $\text{t.d.}_{R/P\cap R}(T/P) = 0$ for all prime ideals P of T; i.e., $R \subseteq T$ is residually algebraic. By similar reasoning, $R \subseteq A$ is residually algebraic for each intermediate ring A. Thus, (R,T) is a residually algebraic pair and, hence, an INC-pair. Since (R,T) is an LO-pair by assumption, an application of the Folklore Theorem completes the proof. □

One cannot delete "LO" from the hypotheses of Proposition 2.5. To see this, consider a ring extension such as $\mathbb{Z} \subseteq \mathbb{Z}_{2\mathbb{Z}}$, and apply the Krull-Akizuki Theorem. Indeed, according to one version of the Krull-Akizuki Theorem (cf. [16, Theorem 93]), if R is a one-dimensional Noetherian domain and T is any overring of R other than $\text{qf}(R)$, then (R,T) is a 1-dimensional pair. For the special case in which R is a principal ideal domain (and, hence, a QR-domain), it is easy to see that $R \neq T$ implies that (R,T) is not an LO-pair. We proceed to generalize these comments by replacing "Noetherian" with "Jaffard". This will produce new examples of 1-dimensional pairs of domains which are not LO-pairs: see also the next Proposition and the comment which follows its proof.

Proposition 2.6. *Let R be a one-dimensional Jaffard domain and T an overring of R which is strictly contained in qf(R). Then (R,T) is a 1-dimensional pair.*

Proof. It suffices to prove that $\dim(A) = 1$ for each ring A such that $R \subseteq A \subseteq T$. Note that $\dim(A) \leq \dim_v(R) = \dim(R) = 1$ (cf. [13, Theorem 6.7.2 (1)]). As A is a domain but not a field, we also have $\dim(A) \geq 1$. Thus, $\dim(A) = 1$. □

In the preceding Proposition, the 1-dimensional pair (R,T) is an INC-pair but need not be an LO-pair, even if R is Noetherian. For example, take $R := \mathbb{Z}$ and $T := \mathbb{Z}_{p\mathbb{Z}}$, for any prime number p.

The above material leads naturally to the question whether there exists an n-dimensional pair which is neither an INC-pair nor an LO-pair. We proceed to give an affirmative answer: see Example 2.8 for the case $n = 2$ and then apply Proposition 2.9 to handle the case $3 \leq n < \infty$. The first step is to isolate (in Example 2.7) some of the properties of a Jaffard domain pair that was constructed by Ben Nasr.

Example 2.7. *(Ben Nasr* [2, Example 3.4]*) Let S be a two-dimensional Prüfer domain with a height 1 maximal ideal M. Consider a subfield D of the residue field $F := S/M$ such that $t.d._D(F) = 1$. (For instance, $S = V_1 \cap V_2$, where $V_1 := K(X)[Y]_{(Y)}$, $V_2 := K(Y) + XK(Y)[X]_{(X)}$, and $D := K$). Let $\varphi : S \to F$ be the canonical surjection and $R := \varphi^{-1}(D)$. Then (R, S) is a 2-dimensional pair.*

Example 2.8. *The 2-dimensional pair (R, S) in Example 2.7 is neither an INC-pair nor an LO-pair.*

Proof. According to [2, Example 3.4], (R, S) is not a residually algebraic pair, and so (R, S) is not an INC-pair. It remains to show that (R, S) is not an LO-pair. Since LO-pairs are GU-pairs [9, Corollary 3.2], it suffices to find a ring T such that $R \subseteq T \subseteq S$ and $T \subset S$ does not satisfy GU. Using the notation in Example 2.7, choose any element $W \in F$ that is transcendental over D, and consider $T := \varphi^{-1}(D[W])$. Choose any nonzero prime ideal P of $D[W]$. Consider the inclusion of prime ideals $M = \varphi^{-1}(0) \subset \varphi^{-1}(P) =: Q$ of T. As $M \in \text{Max}(S)$ and $M \cap T = M$, there is no prime ideal of S that contains M and meets T in Q. It follows that $T \subset S$ does not satisfy GU, to complete the proof. □

Another consequence of the construction in Example 2.7 is that the class of dimensional pairs of domains is not closed under the formation of factor domains. More precisely, if (A, B) is an n-dimensional pair of domains and $J \in \text{Spec}(B)$ with $I := J \cap A$, then the pair of domains $(A/I, B/J)$ need not be a k-dimensional pair for any k. (This behavior of dimensional pairs of domains should be contrasted with that of the classes of LO-pairs, of INC-pairs, and of integral (extension) pairs, each of which is stable under the formation of factor domains (cf. [9, Lemma 3.1 (b)]).) For a proof (with $n = 2$), using the notation of Example 2.7, take $(A, B) := (R, S)$ and $J := M$. Then $(A/I, B/J) = (D, F)$ is not a dimensional pair since the one-dimensional ring $D[W]$ is intermediate between the (zero-dimensional) fields D and F.

We can generate the promised higher-dimensional examples with the following result. The context of Proposition 2.9 arises naturally as an analogue of that of Example 2.3.

Proposition 2.9. *Let V be a finite-dimensional valuation domain of the form $F + M$, where F is a field and M is the maximal ideal of V. Let $R \subseteq T$ be subrings of F. Let $n := \dim(R)$ and $m := \dim(V) < \infty$. Then $(R+M, T+M)$ is an $(n+m)$-dimensional pair $\iff$ (R, T) is an n-dimensional pair. Furthermore,*

$(R+M, T+M)$ is an INC-pair (resp., an LO-pair) if and only if (R,T) is an INC-pair (resp., an LO-pair).

PROOF. If $M = 0$, all the assertions are trivial, and so we assume henceforth that $M \neq 0$. The "LO-pair" part of the "Furthermore" assertion was established in [9, Lemma 2.11 (a)] by using the well known characterizations of the rings intermediate between two classical $D+M$ constructions and of the prime ideals of such rings. The "INC-pair" part of the "Furthermore" assertion may be established in a similar way. For an alternate proof of the "Furthermore" assertion that uses more general pullback-theoretic ideas, one can appeal to [4, Proposition 6], which applies here since $R+M$ and $T+M$ share the nonzero ideal M.

Suppose next that (R,T) is an n-dimensional pair. If S is any ring such that $R+M \subseteq S \subseteq T+M$, then $S = A+M$ for some ring A such that $R \subseteq A \subseteq T$. By the theory of the classical $D+M$ construction [14, Appendix 2], $\dim(A+M) = \dim(A) + \dim(V) = n+m$. Hence, $(R+M, T+M)$ is an $(n+m)$-dimensional pair. (Notice that the proof of this implication did not use the hypothesis that V is finite-dimensional.)

Finally, suppose that $(R+M, T+M)$ is an $(n+m)$-dimensional pair. Let A be any ring such that $R \subseteq A \subseteq T$. As $R+M \subseteq A+M \subseteq T+M$, it follows that $\dim(A+M) = n+m$. But $\dim(A+M) = \dim(A)+\dim(V) = \dim(A)+m$. Since $m < \infty$, we can conclude that $\dim(A) = n$, and so (R,T) is an n-dimensional pair, to complete the proof. □

Proposition 2.9 also has the following consequence. For each integer $n \geq 2$, there exists an n-dimensional LO-pair of domains which is not integral (i.e, which is not an INC-pair). Indeed, the case $n = 2$ is handled by Example 2.4, and then the case $n > 2$ follows via Proposition 2.9.

Although our main purpose for Proposition 2.9 is to obtain higher-dimensional examples, it is interesting to note that there are more general pullback-theoretic versions of its first assertion. One such states the following. Let (V, M) be a quasilocal ring of (Krull) dimension $m < \infty$, put $F := V/M$, let $\varphi : V \to F$ be the canonical surjection, let $D \subseteq E$ be subrings of F, put $R := \varphi^{-1}(D)$ and $T := \varphi^{-1}(E)$, and let $n := \dim(D)$. Then (R,T) is an $(n+m)$-dimensional pair $\iff$ (D,E) is an n-dimensional pair. The proof has the same tempo as the proof of the corresponding part of Proposition 2.9, with the pivotal step "$\dim(A+M) = \dim(A) + \dim(V)$" of the latter being replaced by an appeal to [12, Proposition 2.1 (5)].

In contrast to the comment following the proof of Example 2.8, we pause to record one way in which the behavior of the class of n-dimensional pairs resembles that of the more familiar classes. Consider a ring extension $R \subseteq T$. The ring R_{red} can be viewed canonically as a subring of T_{red} since $\mathrm{Nil}(T) \cap R = \mathrm{Nil}(R)$. According to [5, Proposition 1.15], (R, T) is an LO-pair if and only if (R_{red}, T_{red}) is an LO-pair. Also, according to [5, Proposition 1.15], (R, T) is an INC-pair if and only if (R_{red}, T_{red}) is an INC-pair. Therefore, an application of the Folklore Theorem yields that $R \subseteq T$ is an integral extension if and only if $R_{red} \subseteq T_{red}$ is an integral extension; that is, (R, T) is an integral pair if and only if (R_{red}, T_{red}) is an integral pair. We provide another proof of this fact in part (b) of the next result, while part (a) of that result establishes similar behavior for the class of n-dimensional pairs.

Proposition 2.10. *Let $R \subseteq T$ be a ring extension. Then:*

(a) (R, T) is an n-dimensional pair $\iff$ (R_{red}, T_{red}) is an n-dimensional pair.

(b) $R \subseteq T$ is an integral extension $\iff$ $R_{red} \subseteq T_{red}$ is an integral extension.

Proof. (a) (This proof is similar to that of [5, Proposition 1.12].) As above, we may view R_{red} as a subring of T_{red}. For the "$\Rightarrow$" direction, let A be any ring such that $R_{red} \subseteq A \subseteq T_{red}$. We can write $A = S/\mathrm{Nil}(T)$ for a uniquely determined ring S such that $R + \mathrm{Nil}(T) \subseteq S \subseteq T$. Since (R, T) is assumed to be an n-dimensional pair, we have $\dim(S) = n$. Therefore, the proof of the "$\Rightarrow$" direction concludes by noting that $\dim(S) = \dim(S/\mathrm{Nil}(T)) = \dim(A)$. For the converse, let S be any ring such that $R \subseteq S \subseteq T$. Then $S_{red} := S/\mathrm{Nil}(S) \cong (S + \mathrm{Nil}(T))/\mathrm{Nil}(T)$ satisfies $R_{red} \subseteq S_{red} \subseteq T_{red}$ canonically. Since (R_{red}, T_{red}) is assumed to be an n-dimensional pair, we have $n = \dim(S_{red}) = \dim(S)$, as desired.

(b) One proof of this assertion was given above. We next give a second proof of it. By combining the reasoning in the proof of (a) with a well known result on pullbacks [12, Corollary 1.5 (5)], we see that $R_{red} \subseteq T_{red}$ is an integral extension if and only if $R + \mathrm{Nil}(T) \subseteq T$ is an integral extension. However, the latter condition is easily seen to be equivalent to $R \subseteq T$ being an integral extension, since each element of $\mathrm{Nil}(T)$ is integral over R. □

When seeking n-dimensional pairs that are not integral, we can, as a result of Proposition 2.10, restrict our attention to reduced rings.

Our earlier results concerning Jaffard (or residually Jaffard) domains involved additional hypotheses of either LO-pairs or Krull dimension at most 2. The next result avoids these extra assumptions. It gives an extension of the result [13, Theorem 6.7.2(2)] that for $R \subseteq T$ an integral extension of domains, R is a Jaffard domain if and only if T is a Jaffard domain.

Proposition 2.11. *Let (R,T) be an n-dimensional pair of domains with $n < \infty$. Then (R,T) is a Jaffard domain pair if and only if R is a Jaffard domain.*

Proof. The "only if" assertion is trivial. Conversely, suppose that (R,T) is an n-dimensional pair of domains with $n < \infty$ and R a Jaffard domain. By [6, Proposition 2.2], $\text{t.d.}_R(T) = 0$. Therefore, by [6, Proposition 2.3], $\dim_v(A) \leq n$ for each intermediate domain A. But $\dim_v(A) \geq \dim(A) = n$. Thus, for each intermediate domain A, we have $\dim_v(A) = n = \dim(A) < \infty$; i.e., A is a Jaffard domain. Hence, (R,T) is a Jaffard domain pair. □

Proposition 2.12 stands in contrast to Examples 2.3 and 2.4, by developing a family of examples of INC-pairs which are not n-dimensional pairs. For perhaps the most familiar illustration of Proposition 2.12, consider any finite-dimensional valuation domain R and take T to be any proper overring of R.

Proposition 2.12. *Let (R,T) be an INC-pair of domains such that R is quasilocal and integrally closed in T. If $\dim(R) < \infty$ and $R \neq T$, then (R,T) is not an n-dimensional pair.*

Proof. Deny; i.e., suppose that (R,T) is an n-dimensional pair. In particular, $\dim(R) = \dim(T)$. Then, by [1, Corollary 3.6] (which applies since (R,T) is a residually algebraic pair), the set of rings intermediate between R and T has cardinality $\dim(R) - \dim(T) + 1 = 1$. Hence, $R = T$, the desired contradiction. □

We turn next to ∞-dimensional pairs. They behave somewhat differently than n-dimensional pairs for finite n. For instance, the "$\Rightarrow$" direction of the first equivalence in Proposition 2.9 would fail if $m = \infty$. (To see this, just consider any example that satisfies the hypotheses of Proposition 2.9 and features rings $R \subset T$ such that $\dim(R) \neq \dim(T)$.) There is a wide variety of ∞-dimensional pairs. For instance, the Introduction recalled several examples from [15] of rings R for which (R,T) is an ∞-dimensional pair for each extension ring T of R. The next three results present additional examples of ∞-dimensional pairs.

Proposition 2.13. *Let (R,T) is an LO-pair of rings. Then (R,T) is an ∞-dimensional pair $\iff \dim(R) = \infty$.*

Proof. This is an easy consequence of the fact that LO-pairs are GU-pairs [9, Corollary 3.2]. □

Using methods from [10], we next present two examples of (non-integral) ∞-dimensional pairs in which the intermediate rings have prime ideal chains of arbitrarily large infinite cardinality. In the first (resp., second) example, the maximum cardinality of such a chain is (resp., is not) the same for each of the intermediate rings. In these two results, we assume (as [10] does) the Generalized Continuum Hypothesis (GCH) and, hence, the Axiom of Choice.

Example 2.14. *Let I be an infinite set, let $\{Y_i\}_{i\in I}$ be a set of algebraically independent indeterminates over $\mathbb{Q}$, suppose that X is an indeterminate over $\mathbb{Q}$ which is algebraically independent of $\{Y_i\}$, and put $R := \mathbb{Z}[2X, \{Y_i\}_{i\in I}]$ and $T := \mathbb{Q}[X, \{Y_i\}_{i\in I}]$. Then (R, T) is an ∞-dimensional pair with $R \neq T$ where each intermediate ring has a prime ideal chain of cardinality $2^{|I|}$ but no intermediate ring has a prime ideal chain of cardinality greater than $2^{|I|}$.*

Proof. Since $X \in T \setminus R$, it is clear that $R \neq T$. The fact that each intermediate ring has a prime ideal chain of cardinality $2^{|I|}$ can be established as follows. First, use the GCH to construct a chain of cardinality $2^{|I|}$ consisting of subsets of $\{X\} \cup \{Y_i \mid i \in I\}$ [10, Proposition 2.7]. Next, as in [10, Remark 2.8 (a)], use a degree argument to show that the corresponding ideals of T form a prime ideal chain of cardinality $2^{|I|}$ (with any two distinct subsets in the chain generating distinct ideals). Finally, use a degree argument to show that distinct members of the latter chain meet R (and, hence, meet any intermediate ring) in distinct prime ideals. Therefore, (R, T) is an ∞-dimensional pair.

It remains to show that no intermediate ring has a prime ideal chain of cardinality greater than $2^{|I|}$. This, in turn, holds, as the usual arithmetic of infinite cardinal numbers (that follows from the Axiom of Choice) ensures that $|T| = |I|$. □

Example 2.15. *Let K be a field, let I and J be disjoint infinite sets such that $|I| < |J|$, and put $R := K[\{X_i\}_{i\in I}, \{Y_j\}_{j\in J}]$ and $T := K(\{Y_j\}_{j\in J})[\{X_i\}_{i\in I}]$, where $\{X_i\}_{i\in I} \cup \{Y_j\}_{j\in J}$ is a set of algebraically independent indeterminates over K. Then (R, T) is an ∞-dimensional pair where R has a prime ideal chain of greater cardinality than that of any chain of prime ideals in T.*

Proof. As in the proof of Example 2.14, the methods of [10] lead to a prime ideal chain of $K[\{X_i\}_{i\in I}]$ of cardinality $2^{|I|}$, each of whose members is generated by a subset of $\{X_i\}_{i\in I}$. Any two distinct members of this chain generate distinct prime ideals of T. One concludes easily that each ring intermediate between R and T has a prime ideal chain of cardinality $2^{|I|}$. In particular, (R, T) is an ∞-dimensional pair. Next, note via [10, Remark 2.8 (a)] that R has a prime ideal chain of cardinality $2^{|I\cup J|} = 2^{|J|}$. However, [10, Lemma 2.5] ensures that each prime ideal chain of T has cardinality at most $2^{|I|}$. Since $2^{|J|} > 2^{|I|}$, the proof is complete. □

We conclude with two miscellaneous results. The first of these gives a sufficient condition for n-dimensional pairs that involves the maximal non-Jaffard subrings of a field, a concept which was characterized in [3].

Proposition 2.16. *Let R be a maximal non-Jaffard subring of its quotient field. Let V be a valuation overring of R such that $\dim(V) = \dim_v(R)$, and let T be a subring of V that strictly contains R. Then (T, V) is an n-dimensional pair, where $n := \dim(T)$.*

Proof. It suffices to show that $\dim(S) = \dim(V)$ for each ring S such that $T \subseteq S \subseteq V$. Since V is a valuation overring of S, we have $\dim_v(S) \geq \dim(V)$. But $\dim_v(S) \leq \dim(V)$, since otherwise, $\dim_v(R) \geq \dim_v(S) > \dim(V) = \dim_v(R)$, a contradiction. Thus, $\dim_v(S) = \dim(V)$. But S is a Jaffard domain, by the maximality of R, and so $\dim(S) = \dim_v(S) = \dim(V)$. □

If $R \subset T$ is a ring extension such that no ring S satisfies $R \subset S \subset T$, then T is called a *minimal ring extension* of R; if, in addition, T is an overring of R, then T is called a *minimal overring* of R. It is known that if a domain T is a minimal ring extension of a ring R, then T is an overring of R [18, page 1738, lines 8–13]. Part (b) of our final result shows how a certain type of minimal overring gives rise to an n-dimensional pair. For motivation, note that $\mathbb{Z}[\frac{1}{2}]$ is a non-integral minimal overring of $\mathbb{Z}$ such that $(\mathbb{Z}[\frac{1}{2}] : \mathbb{Z}) = 0$.

Proposition 2.17. *Let T be a minimal ring extension of a ring R. Then:*

(a) T is integral over $R \iff R \subseteq T$ satisfies LO $\iff R \subseteq T$ satisfies GU.

(b) Suppose also that T is a domain, neither R nor T is a field, and $(R : T) = 0$. Then (R, T) is an n-dimensional pair and T is not integral over R.

Proof. (a) Since T is a minimal ring extension of R, integrality of T over R is equivalent to T being a finitely generated R-module. Accordingly, it follows by applying [11, Théorème 2.2 (ii)] to the inclusion map $R \hookrightarrow T$ that T is integral over $R \iff R \subseteq T$ satisfies LO. The equivalence concerning GU now follows by combining the fact that GU $\Rightarrow$ LO [16, Theorem 42] and either the fact that GU-pairs are the same as LO-pairs [9, Corollary 3.2] or the fact that any integral extension satisfies GU (cf. [16, Theorem 44]).

(b) It follows from the proofs of [17, Proposition 3.1(a), Lemme 3.2 and Théorème 3.3] (which implicitly assume that T is not a field) that under the given hypotheses, we have $\dim(R) = \dim(T)$. Since T is a minimal ring extension (in fact, minimal overring) of R, we therefore have that (R, T) is an n-dimensional pair. For the final assertion, it is enough to appeal to the following easy consequence of [11, Proposition 4.1]: any integral minimal ring extension of a nonfield has a nonzero conductor. □

Note that one cannot delete the hypothesis that T is not a field in part (b) of Proposition 2.17 (or, for that matter, in the supporting results on minimal ring extensions that were cited from [17]). For example, consider $R := \mathbb{Z}_{2\mathbb{Z}}$ and $T := \mathbb{Q}$.

References

[1] A. Ayache and A. Jaballah, *Residually algebraic pairs of rings*, Math. Z. **225** (1997), 49–65.

[2] M. Ben Nasr, *Some remarks on residually algebraic pairs of rings*, Arch. Math. (Basel) **78** (2002), 362–368.

[3] M. Ben Nasr and N. Jarboui, *Maximal non-Jaffard subrings of a field*, Publ. Mat. **44** (2000), 157–175.

[4] P.-J. Cahen, *Couples d'anneaux partageant un idéal*, Arch. Math. (Basel) **51** (1988), 505–514.

[5] R. D. Chatham, *Going-down pairs of commutative rings*, Rend. Circ. Mat. Palermo (2) **50** (2001), 509–542.

[6] R. D. Chatham, *On the dimensions of the subrings of a commutative ring*, Int. J. Commut. Rings **2** (2003), 101–105.

[7] R. D. Chatham and D. E. Dobbs, *On open ring pairs of commutative rings*, Houston J. Math. **31** (2005), 65–74.

[8] R. Debremaeker and V. Van Lierde, *Integral ring extensions and LO-pairs*, Comm. Algebra **28** (2000), 2339–2341.

[9] D. E. Dobbs, *Lying-over pairs of commutative rings*, Canad. J. Math. **33** (1981), 454–475.

[10] D. E. Dobbs, *On the prime ideals in a commutative ring*, Canad. Math. Bull. **43** (2000), 312–319.

[11] D. Ferrand and J.-P. Olivier, *Morphismes minimaux d'anneaux*, J. Algebra **16** (1970), 461–471.

[12] M. Fontana, *Topologically defined classes of commutative rings*, Ann.Mat. Pura Appl. **123** (1980), 331–355.

[13] M. Fontana, J. Huckaba and I. J. Papick, *Prüfer Domains*, Dekker, New York, 1997.

[14] R. Gilmer, *Multiplicative Ideal Theory*, Queen's Papers in Pure and Applied Mathematics, **12**, Queen's University Press, Kingston, 1968.

[15] W. J. Heinzer, *Dimensions of extension rings*, in *Zero-dimensional commutative rings*, Lecture Notes in Pure and Appl. Math., **171** (1995), Dekker, New York, 57–64.

[16] I. Kaplansky, *Commutative Rings*, rev. ed., Univ. Chicago Press, Chicago, 1974.

[17] M. Oukessou and A. Miri, *Sur les suranneaux minimaux*, Extracta Math. **14** (1999), 333–347.

[18] J. Sato, T. Sugatani and K. Yoshida, *On minimal overrings of a Noetherian domain*, Comm. Algebra. **20** (1992), 1735–1746.

Received January 4, 2006

(R. Douglas Chatham) Department of Mathematics and Computer Science, Morehead State University, Morehead KY 40351 USA

E-mail address: d.chatham@moreheadstate.edu

(David E. Dobbs) Department of Mathematics, University of Tennessee, Knoxville TN 37996-1300 USA

E-mail address: dobbs@math.utk.edu