Critical Graphs With Respect to Vertex Identification

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Abstract

We explore various types of criticality with respect to differentiating-dominating sets, or identifying codes. Existence and characterization results are included. We conclude with open problems.

Keywords: differentiating-domination, identifying code, critical

1 Introduction

Unless otherwise stated, we follow the terminology of [6]. For any of the graph parameters π discussed here, a π-set is an optimal set with the given property.

The *open neighbourhood* $N(v)$ of a vertex $v \in V_G$ is the set of all vertices adjacent to v. The *closed neighbourhood* $N[v]$ of v is $N[v] = N(v) \cup \{v\}$. For $S \subseteq V$, $\bigcup_{v \in S} N(v)$ is written $N(S)$ and $\bigcup_{v \in S} N[v]$ is written $N[S]$. For $S \subseteq V$, $N_S[v] = N[v] \cap S$ and $N_S(v) = N(v) \cap S$. The *complement* of S is $\overline{S} = V_G - S$.

A set of vertices $S \subseteq V_G$ is a *dominating set* if $N[S] = V_G$. That is, S is a dominating set if every vertex in G is either in S or is adjacent to a vertex in S. The *domination number* $\gamma(G)$ is the smallest cardinality of a dominating set in G.

A set of vertices $L = \{v_1, v_2, ..., v_k\}$ is said to be *locating* if the vector $\overrightarrow{v} = (d(v, v_1), d(v, v_2), ..., d(v, v_k))$ is unique for each vertex v in the graph. In other words, any vertex in the graph can be located if its distance from each of the vertices in the set L is known. This idea of a locating set corresponds to the use of sonar to find objects underwater. The *locating number* of a graph G is defined to be the least number of vertices in a locating set of G.

A dominating set S in a graph G is a *locating-dominating set*, or an LD-set, if for any two vertices v and w in $V_G - S$, $N_S(v) \neq N_S(w)$. The *locating-domination number* $\gamma_L(G)$ is the minimum cardinality of an LD-set in G. For further details on γ_L see [6].

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If a dominating set S in a graph G is such that for any two vertices $v, w \in V_G$, $N_S[v] \neq N_S[w]$, we say S is an *identifying code* or a *differentiating-dominating set*. The *differentiating-domination number* $\gamma_d(G)$ is the minimum cardinality of a differentiating-dominating set if G has such a set, and $\gamma_d(G) = \infty$ otherwise.

Locating-dominating sets were first studied in the context of identifying the location of fires or intruders in a building while differentiating-dominating sets, called identifying codes by coding theorists, arose in determining the location of errors in multiprocessor systems. Identifying codes were introduced in [7]; we use the notation of [4].

Locating-domination involves three-state devices which can distinguish between abnormalities which occur at a vertex and those which occur nearby. Differentiating-dominating sets, on the other hand, rely on more simple two-state devices. The set of processors in a network can be represented by the vertices of a graph while the connections between the processors can be represented by the edges of the graph. A selected subset of the processors can be used to monitor the system by sending periodic error reports to a central monitoring location. Each member of the selected subset of processors sends one of two reports to the central controller: (1) an error has been detected in the neighbourhood of the processor or (2) no errors have been detected. From this information, the controller must be able to determine the exact location of any faulty processor in the system.

An earlier study of related concepts was undertaken by Sumner in [9]. He defined a graph G to be *point-determining* if and only if distinct vertices of G have distinct neighbourhoods. That is, for all v, $w \in V_G$, $N(v) \neq N(w)$. If $\overline{G}$ is point-determining, then G is said to be *point-distinguishing*. Alternately, G is point-distinguishing if and only if $N[v] \neq N[w]$ for all v, $w \in V_G$.

If $N[v] = N[w]$ for two vertices $v, w \in V_G$, we say v and w are *indistinguishable*. In this case, note that $\gamma_d(G) = \infty$. For example, if G is a complete graph of order at least two, then $\gamma_d(G) = \infty$. Graphs for which the differentiating-domination number is finite are said to be *distinguishable*.

Theorem 1 (Gimbel, et. al. [4] and Karpovsky, et. al. [7]) *If G is of order n, then $\gamma_d(G) \geq \lceil \log_2(n+1) \rceil$.*

Theorem 2 ([4]) *Let P_n be the path on n vertices and let C_n be the cycle on n vertices.*

(a) *For $k \geq 2$, $\gamma_d(P_{2k}) = k + 1$.*

(b) *For $k \geq 1$, $\gamma_d(P_{2k+1}) = k + 1$.*

(c) *For $k \geq 3$, $\gamma_d(C_{2k}) = k$.*

(d) *For $k \geq 3$, $\gamma_d(C_{2k+1}) = k + 2$.*

By considering the probability of a pair of indistinguishable vertices in a random graph, it is also shown in [4] that almost every graph is distinguishable. Furthermore, if a graph is not distinguishable, it can be embedded in a distinguishable graph.

Theorem 3 ([4]) *If G is a graph of order n, then G embeds as an induced subgraph in a distinguishable graph of order $n + \lceil \log_2 n \rceil$.*

A lower bound on the differentiating-domination number that includes information about the order of the graph as well as the cardinality of certain vertex neighbourhoods is given as Theorem 1.3 in [7]. Let G be a graph of order n and let $V_i = |N[v_i]|$ for $1 \leq i \leq n$. Assume that the V_i are indexed such that $V_1 \geq V_2 \geq \cdots \geq V_n$.

Theorem 4 ([7]) *Let G be a graph of order n such that $n/2 \geq V_1$. Let K be the smallest integer such that for some l with $1 \leq l \leq \min(K, V_1)$ the two following conditions are satisfied:*

$$n \leq \sum_{j=1}^{l-1} \binom{K}{j} + \left\lfloor \frac{1}{l} \left(\sum_{i=1}^{K} V_i - \sum_{j=1}^{l-1} j \binom{K}{j} \right) \right\rfloor$$

and

$$\sum_{j=1}^{l-1} j \binom{K}{j} < \sum_{i=1}^{K} V_i \leq \sum_{j=1}^{l} j \binom{K}{j}.$$

Then $\gamma_d(G) \geq K$.

Let $K(G)$ be the set of cliques in a graph G. The *clique-graph* $C(G)$ of G is defined in [3] as having the elements of $K(G)$ as vertices with two vertices A and B adjacent if and only if the cliques A and B have a nonempty intersection in G. A graph G is *clique-critical* if its clique graph changes whenever any vertex is removed.

A set $\mathcal{L}$ satisfies the *Helly property* if for any subfamily $\mathcal{L}' \subseteq \mathcal{L}$ with $S_i \cap S_j \neq \emptyset$ for all $S_i, S_j \in \mathcal{L}'$, we have $\cap\{S : S \in \mathcal{L}'\} \neq \emptyset$.

Lim [8] defines a *supercompact graph G* to be the intersection graph of some family $\mathcal{L}$ of subsets of a set X such that $\mathcal{L}$ satisfies the Helly property and for any $x \neq y$ in X, there exists $S \in \mathcal{L}$ with $x \in S$, $y \notin S$.

Theorem 5 (Lim [8]) *A graph G is supercompact if and only if G is distinguishable.*

From the work of Gimbel and Lim, we see that almost every graph is supercompact. In particular, Lim shows every clique-critical graph is supercompact and hence distinguishable.

2 The Differentiating-domination number

Theorem 7 in [4] states that if G is a connected distinguishable graph of order n, then $\gamma_d(G) \leq n - 1$. This result is correct, but the proof in [4] is incorrect. It is shown there that if x is a vertex in G such that $V_G - \{x\}$ is not a differentiating-dominating set of G, then there are two vertices, y and z, in G such that $xy \in E_G$, $xz \notin E_G$, and $N[z] = N[y] - \{x\}$. This is correct, but it is then concluded that $V_G - \{y\}$ is necessarily a differentiating-dominating set of G. This conclusion is incorrect. For example, if G is the path $wxyz$, then $V_G - \{x\}$ is not a differentiating-dominating set of G. There are two vertices, y and z, in with $xy \in E_G$, $xz \notin E_G$, and $N[z] = N[y] - \{x\}$, but $V_G - \{y\}$ is not a differentiating-dominating set either. Furthermore, if $G^* = G + K_1$ then none of $V_{G^*} - \{x\}$, $V_{G^*} - \{y\}$, or $V_{G^*} - \{z\}$ is a differentiating-dominating set in G^*. It soon becomes clear that the result cannot be proved by a simple interchange of vertices. We now provide a proof for the theorem based on the following lemma.

Lemma 6 *Suppose G is a distinguishable graph with $\gamma_d(G) = |V_G|$ and v is a vertex with positive degree in G. Then there exists an induced path $pqrs$ in G such that $v = q$, $N[p] = N[q] - \{r\}$ and $N[s] = N[r] - \{q\}$.*

Proof. If $C = V_G - \{v\}$, then C is not differentiating-dominating, so there are two vertices r and s such that $N[s] \neq N[r]$, but $N_C[s] = N_C[r]$.

Note that neither r nor s is the vertex v, since $N_C[v] = N_C[v']$ implies $N(v) = N[v'] - \{v\}$. But this implies $N[v] = N[v']$ for the vertices v and v', which contradicts the fact that G is distinguishable.

Since $N[s] \neq N[r]$ but $N_C[s] = N_C[r]$, v is adjacent to exactly one of r and s. Without loss of generality say v is adjacent to r, in which case $N_C[s] = N_C[r]$ implies

$$N[s] = N[r] - \{v\}. \tag{1}$$

Similarly, if $D = V_G - \{r\}$, then there are two vertices p and q, neither of which is r, such that q is adjacent to r and

$$N[p] = N[q] - \{r\}. \tag{2}$$

Now $p \neq v$, since v is adjacent to r but p is not.

It now follows from (1) that p is not adjacent to s and hence (2) implies that q is not adjacent to s. Therefore $q \notin N[s]$, so $q \in N[r] - N[s]$. This implies $q = v$.

We have also shown that p is nonadjacent to both r and s, and q is nonadjacent to s. Hence $pqrs$ is an induced path in G with $q = v$. ■

Theorem 7 *Let G be a distinguishable graph with at least one edge. Then $\gamma_d(G) \leq n - 1$.*

Proof. Suppose, to the contrary, that $\gamma_d(G) = n$. We shall show that for every integer $k \geq 2$ there exist k paths $P^1, \ldots, P^k$ such that $V(P^k) \nsubseteq \bigcup_{i=1}^{k-1} V(P^i)$, thus contradicting the finiteness of G.

Let v be a vertex with positive degree in G. By Lemma 6 there is an induced path $P^1 := p^1q^1r^1s^1$ in G with $v = q^1$ and

$$N[p^1] = N[q^1] - \{r^1\} \tag{1a}$$

and

$$N[s^1] = N[r^1] - \{q^1\}. \tag{1b}$$

Now, applying Lemma 6 to the vertex p^1, we obtain an induced path $P^2 := p^2q^2r^2s^2$ with $p^1 = q^2$ and

$$N[p^2] = N[q^2] - \{r^2\} \tag{2a}$$

and

$$N[s^2] = N[r^2] - \{q^2\}. \tag{2b}$$

Since $p^1 = q^2$, we see $p^2 \in N[p^1]$. From (1a), $N[p^1] \subset N[q^1]$, so $p^2 \in N[q^1]$. Thus, p^2 is adjacent to p^1 and is either adjacent to or equal to q^1. Similarly, r^2 is adjacent to p^1, so

$$r^2 \in N[q^1]. \tag{3}$$

By (2a), p^2 is neither adjacent nor equal to r^2, so $p^2 \neq q^1$; hence p^2 is adjacent to q^1, which further implies that $r^2 \neq q^1$. Since p^2 is adjacent to both p^1 and q^1, p^2 is not on the induced path P^1.

Furthermore, by (2b) and (3), s^2 is adjacent to q^1. Since s^2 is not adjacent to p^1, it follows from (1) that $s^2 = r^1$.

Thus, by (2b), r^2 is adjacent to all vertices in P^1. So neither p^2 nor r^2 is on the induced path P^1. Thus

$$p^2, r^2 \notin V(P^1) \text{ and } s^2 = r^1.$$

This proves the case $k = 2$.

Now suppose that for some $k \geq 3$ we have obtained $k-1$ paths $P^1, \ldots, P^{k-1}$ in G with $P^i := p^iq^ir^is^i$ such that

$$N[p^i] = N[q^i] - \{r^i\} \tag{ia}$$

and

$$N[s^i] = N[r^i] - \{q^i\} \tag{ib}$$

for $i = 1, \ldots, k-1$ and

$$p^j, r^j \notin \bigcup_{i=1}^{j-1} V(P^i) \text{ and } q^j = p^{j-1} \text{ and } s^j = r^{j-1}$$

for $j = 2, \dots, k-1$.

For each of the paths $P^1, \dots, P^{k-1}$, colour the first two vertices white and the last two black.

By repeated application of (ia),

$$N[p^{k-1}] \subset N[q^{k-1}] = N[p^{k-2}] \subset N[q^{k-2}] = \cdots = N[p^1] \subset N[q^1].$$

Therefore all the white vertices in $\bigcup_{i=1}^{k-1} V(P^i)$ form a clique. By repeated application of (ib),

$$N[s^1] \subset N[r^1] = N[s^2] \subset N[r^2] = \cdots = N[s^{k-1}] \subset N[r^{k-1}].$$

Since $p^{k-1} \notin N[r^{k-1}]$, this implies p^{k-1} is not adjacent to any black vertex in $\bigcup_{i=1}^{k-1} V(P^i)$.

Now apply Lemma 6 to the vertex p^{k-1} to obtain a path $p^k q^k r^k s^k$. Since $q^k = p^{k-1}$ is not adjacent to any black vertices, neither p^k nor r^k is black. However, since p^k is not adjacent to r^k, it is not the case that both vertices are white. Thus at least one of p^k or r^k is uncoloured and so either $p^k \notin \bigcup_{i=1}^{k-1} V(P^i)$ or $r^k \notin \bigcup_{i=1}^{k-1} V(P^i)$. Therefore $V(P^k) \nsubseteq \bigcup_{i=1}^{k-1} V(P^i)$. ■

3 Criticality Concepts

Following [5], for each parameter π define the simple finite graph G to be

C1 π-*critical* if $\pi(G-v) < \pi(G)$ for all $v \in V_G$,

C2 π^+-*critical* if $\pi(G-v) > \pi(G)$ for all $v \in V_G$,

C3 π-*edge-critical* if $\pi(G+e) < \pi(G)$ for all $e \in E_{\overline{G}}$,

C4 π^+-*edge-critical* if $\pi(G+e) > \pi(G)$ for all $e \in E_{\overline{G}}$,

C5 π-*ER-critical* if $\pi(G-uv) > \pi(G)$ for all $uv \in E_G$,

C6 π^--*ER-critical* if $\pi(G-uv) < \pi(G)$ for all $uv \in E_G$.

For each type of criticality with respect to $\pi = \gamma_d$, we say G is *finitely critical* if γ_d is finite both before and after the corresponding modification to G. For example, if G is a graph with $\gamma_d(G) < \gamma_d(G+e) < \infty$ for all $e \in E_{\overline{G}}$, then we say G is finitely γ_d^+-edge-critical.

See [1, 5, 10] for further results and references regarding criticality with respect to parameters $\pi = \{ir, \gamma, i, \beta, \Gamma, IR\}$. We provide basic existence results for the six types of criticality with $\pi = \gamma_d$.

3.1 γ_d-critical

Any edgeless graph with at least one vertex is γ_d-critical. The following proposition shows some other easy examples of connected graphs that are γ_d-critical.

Proposition 8 *If $G = P_2$, C_4, or $K_{n,m}$ with $n > m \geq 3$, then G is γ_d-critical.*

Proof. Deletion of a vertex from P_2 leaves a singleton. Since $\gamma_d(P_2) = \infty$ and $\gamma_d(K_1) = 1$, P_2 is γ_d-critical.

Deletion of a vertex from C_4 yields P_3. Since $\gamma_d(C_4) = 3$ and $\gamma_d(P_3) = 2$, C_4 is γ_d-critical.

For $n \geq m \geq 2$, $\gamma_d(K_{n,m}) = n + m - 2$. So for $n \geq m \geq 3$, deletion of a vertex reduces the differentiating-domination number by one. ■

Proposition 9 *Odd cycles C_{2k+1} are finitely γ_d-critical for $k \geq 3$.*

Proof. Deletion of any single vertex from C_{2k+1} yields P_{2k}. By Theorem 2, if $k \geq 3$ then $\gamma_d(C_{2k+1}) = k + 2 > k + 1 = \gamma_d(P_{2k})$. ■

Proposition 10 *If $\delta(G) \geq 1$, $\gamma_d(G) = n - 1$, and $G - v$ is distinguishable for each $v \in V_G$, then G is γ_d-critical.*

Proof. This follows immediately from Theorem 7. ■

3.2 γ_d^+-critical

Proposition 11 *There are no γ_d^+-critical graphs.*

Proof. Suppose G is a γ_d^+-critical graph of order n. Let $C \subseteq V_G$ be a γ_d-set in G. Since G is γ_d^+-critical, such a set C exists.

Suppose there exists a vertex $v \notin C$. If v is deleted, then C is differentiating-dominating in $G - v$, although C may no longer be a γ_d-set. This implies that deletion of v either reduces the differentiating-domination number or leaves it unchanged. Therefore, if G is γ_d^+-critical, then $\gamma_d(G) = n$.

By Theorem 7, G is edgeless, so deletion of a vertex reduces the differentiating-domination number. Thus there are no γ_d^+-critical graphs. ■

Corollary 12 *If G is supercompact, then $G - v$ is supercompact for at least one $v \in V_G$.*

Proof. This follows from Theorem 5 and the proof of Proposition 11. ■

3.3 γ_d-edge-critical

Adding an edge to an indistinguishable graph may result in a distinguishable graph. For example, if $G = P_2 \cup K_1$, then $\gamma_d(G) = \infty$, but $\gamma_d(G + e) = 2$ for every $e \in E(G)$.

There also exist finitely γ_d-edge-critical critical graphs.

Proposition 13 *Odd cycles of order $n \geq 7$ are finitely γ_d-edge-critical.*

Proof. Label the vertices of C_{2k+1} as $1, 2, \cdots, 2k + 1$. By symmetry, we need consider only the addition of edges $\{1, x\}$ where $x \in \{3, 4, \cdots, k + 1\}$. If x is even, the set of odd vertices beginning with vertex 3 together with vertex 4 serves to distinguish all vertices in $C_{2k+1} + \{1, x\}$. If x is odd, then the set of odd vertices serves to distinguish all vertices. Thus, $\gamma_d(C_{2k+1} + e) \leq k + 1 < k + 2 = \gamma_d(C_{2k+1})$. ■

However, not all cycles are γ_d-edge-critical.

Proposition 14 *Even cycles C_{2k} are not γ_d-edge-critical for $k \geq 2$.*

Proof. Note that $C_4 + e$ is indistinguishable for all $e \in E_{\overline{G}}$ so C_4 is not γ_d-edge-critical. Furthermore, by Theorem 1, C_6 and C_8 are not γ_d-critical, since $\gamma_d(C_6 + e) \geq \lceil \log {}_2 7 \rceil = 3 = \gamma_d(C_6)$ and $\gamma_d(C_8 + e) \geq \lceil \log {}_2 9 \rceil = 4 = \gamma_d(C_8)$.

For $k \geq 5$, label the vertices of C_{2k} as $1, 2, \cdots, 2k$. By symmetry, we need consider only the addition of edges $\{1, x\}$ where $x \in \{3, 4, \cdots, k+1\}$. Whether x is even or odd, the set of even vertices in $C_{2k} + \{1, x\}$ serves to distinguish all vertices, so $\gamma_d(C_{2k} + e) \leq k$. We show that $\gamma_d(C_{2k} + e) \geq k$ by application of Theorem 4.

From Theorem 4, suppose $K = k - 1$. For $C_{2k} + e$, note that $|N[v]| = 4$ if v is incident with e and $|N[v]| = 3$ otherwise. Thus

$$\sum_{i=1}^{K} V_i = 4 \cdot 2 + 3(K - 2) = 3K + 2 = 3(k-1) + 2 = 3k - 1.$$

Since $1 \leq l \leq \min(K, 4)$, we consider each possible value of l.

Case 1. If $l = 1$, then the second condition on K and l in Theorem 4 is not satisfied since $3k - 1 \not< \sum_{j=1}^{1} j\binom{k-1}{j} = k - 1$.

Case 2. If $l = 2$, then the first condition on K and l is not satisfied since

$$2k \not\leq \sum_{j=1}^{l-1} \binom{k-1}{j} + \left\lfloor \frac{1}{l} \left(3k - 1 - \sum_{j=1}^{l-1} j \binom{k-1}{j} \right) \right\rfloor = k - 1 + \left\lfloor \frac{1}{2}(2k) \right\rfloor = 2k - 1.$$

Case 3. If $l = 3$ then the second condition on K and l is not satisfied. The left inequality implies that $k - 1 + 2\binom{k-1}{2} < 3k - 1$. This implies that $2\binom{k-1}{2} < 2k$ which is not true for $k \geq 5$.

Case 4. Similarly, if $l = 4$ then the second condition on K and l implies $k - 1 + 2\binom{k-1}{2} + 3\binom{k-1}{3} < 3k - 1$. This implies $2\binom{k-1}{2} + 3\binom{k-1}{3} < 2k$ which is again not true for $k \geq 5$. ■

3.4 γ_d^+-edge-critical

There are graphs for which $\gamma_d(G) < \gamma_d(G+e) = \infty$ for all $e \in E_{\overline{G}}$. For example, P_3, C_4 and mK_1, $m \geq 2$, have this property.

There are also γ_d^+-edge-critical graphs for which the increase is finite for some edges but infinite for others. For example, if G is the graph in Figure 1, then $\gamma_d(G) = 3$ and $\gamma_d(G + c_1 c_3) = 4$, while $\gamma_d(G + c_1 c_2) = \infty$.

If G is a finitely γ_d^+-edge-critical graph, then its differentiation-domination number will be at most $|V_G| - 3$. In order to prove this, we need the following lemma.

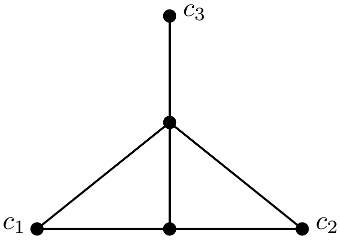


Figure 1: A γ_d^+-edge-critical graph.

Lemma 15 *Suppose G is a finitely γ_d^+-edge-critical graph and C is a γ_d-set of G. Then the subgraph of G induced by $\overline{C}$ is a clique.*

Proof. Suppose there are two nonadjacent vertices v and w in $\overline{C}$. Then C is clearly a γ_d-set of $G + vw$, so $\gamma_d(G + vw) \leq |C| = \gamma_d(G)$. But then G is not γ_d^+-edge-critical. ■

Theorem 16 *Suppose G is a connected finitely γ_d^+-edge-critical graph. Then $\gamma_d(G) \leq |V_G| - 3$.*

Proof. It follows from Theorem 7 that $\gamma_d(G) \neq |V_G| - 1$.

Suppose $\gamma_d(G) = |V_G| - 2$ and that $C = V_G - \{x_1, x_2\}$ is a γ_d-set of G. By Lemma 15, x_1 and x_2 are adjacent. Let $B = N(x_1) \cap N(x_2)$ and for $i = 1, 2$, $A_i = (N(x_i) \cap C) - B$. Either A_1 or A_2 is nonempty, otherwise C would not distinguish between x_1 and x_2. Without loss of generality let $y_1 \in A_1 = (N(x_1) \cap C) - N(x_2)$.

Since $\gamma_d(G) < \gamma_d(G + x_2y_1)$, there is a vertex y_2 such that $N_C[x_2] \cup \{y_1\} = N_C[y_2]$, else C would be a differentiating-dominating set in $G + x_2y_1$. Note that y_2 is adjacent to both x_2 and y_1. The vertex x_1 is adjacent to both x_2 and y_1, so if $y_2 \in N[x_1]$, then $N_G[x_2] \cup \{y_1\} = N_G[y_2]$ and $G + x_2y_1$ is indistinguishable. Hence $y_2 \in A_2$.

Since $N_C[x_2] = N_G(x_2) - \{x_1\}$,

$$(N_G[x_2] - \{x_1\}) \cup \{y_1\} = N_C[y_2] \cup \{x_2\},$$

which implies

$$N_G[x_2] - \{x_1\} = N_G[y_2] - \{y_1\}. \tag{1}$$

Similarly, in $G + x_1y_2$ there is a vertex w such that $N_G[x_1] - \{x_2\} = N_G[w] - \{y_2\}$. Since w is adjacent to y_2 but not to x_2, and we have not made any assumptions on $N(y_1)$ other than $x_1, y_2 \in N(y_1)$ and $x_2 \notin N(y_1)$, we may assume that $y_1 = w$. Hence

$$N_G[x_1] - \{x_2\} = N_G[y_1] - \{y_2\}. \tag{2}$$

Let $C^* = V_G - \{x_1, y_2\}$. Since x_1 is not adjacent to y_2, it follows from Lemma 15 that C^* is not a γ_d-set in G. Hence there are two vertices u and v such that

$$N_{C^*}[u] = N_{C^*}[v]. \tag{3}$$

Note that C^* is obtained from C by interchanging y_2 and x_2. We consider three cases.

Case 1. Suppose $u, v \notin \{x_1, y_1\}$. Let $z \in N_C[u]$.

If $z \neq y_2$, then $z \in C^*$ and hence $z \in N_{C^*}[u] = N_{C^*}[v]$ by (3). Since $z \in C$ by our assumption, it follows that $z \in N_C[v]$.

If $z = y_2$ then it follows from (1) that x_2 is a neighbour of u. Since $x_2 \in C^*$, it follows from (3) that x_2 is a neighbour of v. But then it follows from (1) that y_2 is a neighbour of v. Hence $y_2 \in N_C[v]$. This proves that $N_C[u] \subseteq N_C[v]$. The proof that $N_C[v] \subseteq N_C[u]$ is similar. Hence $N_C[u] = N_C[v]$, contradicting our assumption that C is a γ_d-set.

Case 2. If $u = x_1$, then $N_{C^*}[x_1] = N_{C^*}[v]$ by (3). Note that $N_{C^*}[x_1] = N_G[x_1] - \{x_1\}$, so (2) and (3) imply that

$$\begin{aligned} N_{C^*}[x_1] &= ((N_G[y_1] - \{y_2\}) \cup \{x_2\})) - \{x_1\} = (N_G[y_1] - \{y_2, x_1\}) \cup \{x_2\} \\ &= N_{C^*}[y_1] \cup \{x_2\}. \end{aligned}$$

Application of (3) thus implies

$$N_{C^*}[v] = N_{C^*}[y_1] \cup \{x_2\}. \tag{4}$$

If $v \in C^*$, then $N_{C^*}[v] = N_G[v] - x_1$, and (4) implies

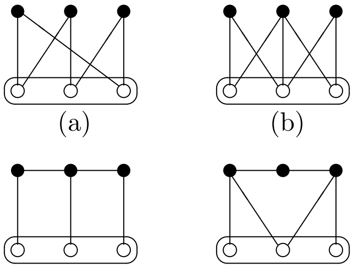
$$N_G[v] - \{x_1\} = N_{C^*}[y_1] \cup \{x_2\} = (N_G[y_1] - \{y_2, x_1\}) \cup \{x_2\},$$

which implies $N_G[v] = N_G[x_1]$ by (2). However, this contradicts our assumption that G is distinguishable.

Therefore $v \notin C^*$, so $v = y_2$. Recall that $y_2 \in A_2$, so y_2 is not adjacent to x_1. Since $N_{C^*}[x_1] = N_G(x_1)$ and $N_{C^*}[y_2] = N_G(y_2)$, (3) implies $N_G(x_1) = N_G(y_2)$. Therefore $N_{G+x_1y_2}[x_1] = N_{G+x_1y_2}[y_2]$, hence $\gamma_d(G + x_1y_2) = \infty$, contradicting our assumption that G is finitely γ_d^+-edge-critical.

Case 3. If $u = y_1$, then (3) implies $N_{C^*}[y_1] = N_{C^*}[v]$ for some vertex v. Since $N_{C^*}[y_1] = N_G[y_1] - \{x_1, y_2\} = N_G(x_1) - \{x_2\}$ by (2), $N_{C^*}[v] = N_G(x_1) - \{x_2\}$. This implies that v is not adjacent to x_2. Furthermore, by (1), v is not adjacent to y_2.

If v is not adjacent to x_1, then by (2) v also is not adjacent to y_1. This contradicts the fact that $N_{C^*}[y_1] = N_{C^*}[v]$, so v is adjacent to x_1. Therefore $N_G[v] = N_G[x_1] - \{x_2\} = N_G[y_1] - \{y_2\}$, which implies $N_{G+vy_2}[v] = N_{G+vy_2}[y_1]$. Thus $\gamma_d(G + vy_2) = \infty$, again contradicting our assumption that G is finitely γ_d^+-edge-critical. ■



(a) (b)

(c) (d)

Figure 2: Four graphs with $n = 6$ and $\gamma_d(G) = 3$.

Figure 3: Two graphs with $n = 7$ and $\gamma_d(G) = 3$.

Proposition 17 *If G is a connected finitely γ_d^+-edge-critical graph, then $\gamma_d(G) \geq 4$.*

Proof. Suppose to the contrary that G is a connected finitely γ_d^+-edge-critical graph of order n with $\gamma_d(G) < 4$. Let C be a γ_d-set in G.

By Theorem 1, we see that if $\gamma_d(G) = 1$, then $n = 1$. Furthermore, if $\gamma_d(G) = 2$ then either $n = 2$ or $n = 3$, and if $\gamma_d(G) = 3$ then $3 \leq n \leq 7$. By Theorem 16, $\gamma_d(G) + 3 \leq n$. Thus, it must be the case that $\gamma_d(G) = 3$ and either $n = 6$ or $n = 7$.

Note the subgraph induced by C is either a path of length two or consists of three isolates and, by Lemma 15, the subgraph induced by $\overline{C}$ is complete. Figure 2 shows the four possible graphs for $n = 6$ and Figure 3 shows the two possible graphs for $n = 7$.

None of these six graphs is finitely γ_d^+-edge-critical. In Figure 2(a) and 2(c), an edge can be added that leaves the differentiating-domination number unchanged. In all the other cases, an edge can be added that results in an indistinguishable graph. ■

We now describe a construction of a finitely γ_d^+-edge-critical graph.

Construction 1 *To construct a finitely γ_d^+-edge-critical graph G of order 11 with $\gamma_d(G) = 4$:*

1. *Begin with C, a copy of C_4, and K, a copy of K_7.*
2. *Label the vertices of C as $1, 2, 3,$ and 4, and label the vertices of K as $a, b, c, d, e, f,$ and g.*

3. *Add edges from a to $\{1,2,3,4\}$, from b to $\{1,2\}$, from c to $\{1,3\}$, from d to $\{1,4\}$, from e to $\{2,3\}$, from f to $\{2,4\}$, and from g to $\{3,4\}$.*

Note that C is a γ_d-set of order 4 in G. We claim that G is finitely γ_d^+-edge critical.

By symmetry, there are three edges to consider: $13, 2c$, and $3b$. We claim that addition of any of these edges yields a distinguishable graph G^* with $4 < \gamma_d(G^*) < \infty$.

Lemma 18 *The graph G^* is distinguishable.*

It can be seen that $N[1] \neq N[3]$ after the addition of the edge 13, and that $N[1] \neq N[v]$ and $N[3] \neq N[v]$ for any $v \in K$. Similarly, $N[2] \neq N[c]$ and $N[3] \neq N[b]$ after the other modifications. However, C no longer serves as a γ_d-set under any of these modifications.

Note that vertex a is adjacent to all the vertices in G or G^* and is therefore not an element of any γ_d-set.

Observation 19 *Four vertices from K cannot distinguish G^* since they do not distinguish the vertices in K.*

Lemma 20 *One vertex from C and three vertices from K cannot distinguish G^*.*

Proof. Let $C_1^* = \{c_1, k_1, k_2, k_3\}$ for some $c_1 \in C$ and $k_i \in K$. By construction, vertex c_1 is adjacent to exactly four vertices from K in G, while in G^* the vertex c_1 is adjacent to either four or five vertices in K. Thus, there are at least three vertices in K which cannot be distinguished from a by C_1. ■

Lemma 21 *Two vertices from C and two vertices from K cannot distinguish G^*.*

Proof. Let $C_2^* = \{c_1, c_2, k_1, k_2\}$ for $c_i \in C$ and $k_i \in K$. By construction, there is at least one vertex $v \in K$ with $v \neq a$ such that v is adjacent to both c_1 and c_2. Therefore, v cannot be distinguished from a by C_2. ■

Lemma 22 *Three vertices from C and one vertex from K cannot distinguish G^*.*

Proof. Let $C_3^* = \{c_1, c_2, c_3, k\}$. We consider two cases.

Suppose the edge 13 is added. If k is adjacent to 1, then C_3^* does not distinguish between vertices 1 and a. If k is adjacent to 3, then C_3^* does not distinguish between 3 and a. If k is adjacent to neither 1 nor 3, then $k = f$ and thus 1 and 3 are not distinguishable.

Now suppose either of the other two edges is added.

Suppose vertex 1 is switched with a vertex $k \in K$ to yield C_3^*. If k is c, e, or g then vertices 3 and a are not distinguishable by C_3^*. Furthermore, if k is b or d, then vertices 1 and f are not distinguishable. Finally, if k is vertex f, then vertices 2 and e are not distinguishable. Therefore, $C_3^* \neq \{2,3,4,k\}$ for any $k \in K$.

Similarly, it is straightforward to see that none of $\{1,3,4,k\}$, $\{1,2,4,k\}$, or $\{1,2,3,k\}$ can serve to distinguish G for any $k \in K$. $\blacksquare$

Therefore, the addition of any edge to G yields a distinguishable graph while increasing the differentiating-domination number. This proves the following proposition.

Proposition 23 *The graph constructed in Construction 1 is finitely* γ_d^+*-edge-critical.*

3.5 γ_d-ER-critical

Consider a graph G with $\gamma_d(G) < \infty$ but for which $G - e$ is not distinguishable for any $e \in E_G$. Entringer and Gassman [2] called graphs with this property *line-critical point distinguishing.* They showed that the only connected nontrivial line-critical point distinguishing graph is the path on three points.

Theorem 24 (Entringer and Gassman [2]) *A graph has* $\gamma_d(G) < \infty$ *and* $\gamma_d(G-e) = \infty$ *for all* $e \in E_G$ *if and only if it is the union of isolated vertices and disjoint paths of length two.*

Proposition 25 *Even cycles* C_{2k} *are finitely* γ_d*-ER-critical for* $k \geq 3$.

Proof. Deletion of any single edge from C_{2k} yields P_{2k}. By Theorem 2, if $k \geq 3$ then $\gamma_d(C_{2k}) = k < k+1 = \gamma_d(P_{2k})$. $\blacksquare$

3.6 γ_d^--ER-critical

Removing an edge can cause the differentiating-domination number to decrease from the infinite to a finite value.

Proposition 26 *Odd cycles* C_{2k+1} *are finitely* γ_d^-*-ER-critical for* $k \geq 3$.

Proof. Deletion of any single edge from C_{2k+1} yields P_{2k+1}. By Theorem 2, if $k \geq 3$ then $\gamma_d(C_{2k+1}) = k+2 > k+1 = \gamma_d(P_{2k+1})$. $\blacksquare$

4 Open Questions

This is the first study of criticality with respect to differentiating-domination, so numerous open questions remain.

1. Construction 1 can be extended to yield a finitely γ_d^+-edge-critical graph G with $\gamma_d(G) = k$ for each even k greater than or equal to 4. Does there exist a finitely γ_d^+-edge-critical graph which has a γ_d-set of odd order?

2. The maximum cardinality of a minimal locating-dominating set in a graph G is called the *upper locating-domination number* and is denoted $\Gamma_L(G)$. The *upper differentiating-domination number* of G, $\Gamma_d(G)$, is the maximum cardinality of a minimal differentiating-dominating set in G. Consider criticality for γ_L, Γ_L, and Γ_d.

3. Consider noncritical graphs, in which, say, any edge could be deleted without changing the value of the parameter in question. Edge-deletion noncriticality seems to be related to fault-tolerant networks. What would noncriticality mean in terms of other graph alterations?

4. We define *pseudocritical* graphs to be graphs in which every deletion/addition causes the parameter to change, but not necessarily in the same direction. For example, the path of length 3, with $\gamma_d(P_4) = 3$, is *vertex pseudocritical.* Deletion of an end vertex causes γ_d to decrease to 2 while deletion of an interior vertex causes γ_d to increase to ∞. Are there pseudocritical graphs for which the differentiation-domination number remains finite after each alteration?

5. We define *extracritical* graphs to be graphs which remain critical after modification. Suppose, for example, that G is γ_d-critical and G' is the result of a vertex deletion. The graph G is *Type I extracritical* if G' is also γ_d-critical. If G' is not γ_d-critical but is, say, γ_d-ER-critical, then G is *Type II extracritical.* If an extracritical graph G is π-critical for some parameter π, what is known about the modified graph G'?

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References

[1] E. J. Cockayne, O. Favaron, and C. M. Mynhardt, Irredundance-edge-removal-critical graphs, *Discrete Math.* **276** (2004) 111−125.

[2] R. C. Entringer and L. D. Gassman, Line-critical point determining and point distinguishing graphs, *Discrete Math.* **10** (1974) 43−55.

[3] F. Escalante and B. Toft, On clique-critical graphs, *J. Combinatorial Theory Ser. B* **17** (1974) 170−182.

[4] J. Gimbel, B. D. van Gorden, M. Nicolescu, C. Umstead, and N. Vaiana, Location with dominating sets, *Congr. Numer.* **151** (2001) 129−144.

[5] P. J. P. Grobler and C. M. Mynhardt, Upper domination parameters and edge critical graphs, *J. Combin. Math. Combin. Comput.* **33** (2000) 239−251.

[6] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater, *Fundamentals of Domination in Graphs.* Marcel Dekker, New York, 1998.

[7] M. G. Karpovsky, K. Chakrabarty, and L. B. Levitin, On a new class of codes for identifying vertices in graphs, *IEEE Trans. Inform. Theory* **44(2)** (1998) 599−611.

[8] C. K. Lim, On supercompact graphs, *J. Graph Theory* **2** (1978) 349−355.

[9] D. P. Sumner, Point determination in graphs, *Discrete Math.* **5** (1973) 179−187.

[10] D. P. Sumner and E. Wojcicka, Graphs critical with respect to the domination number, in T. W. Haynes, S. T. Hedetniemi and P. J. Slater (Eds.), *Domination in Graphs: Advanced Topics*: 439−469. Marcel Dekker, New York, 1998.